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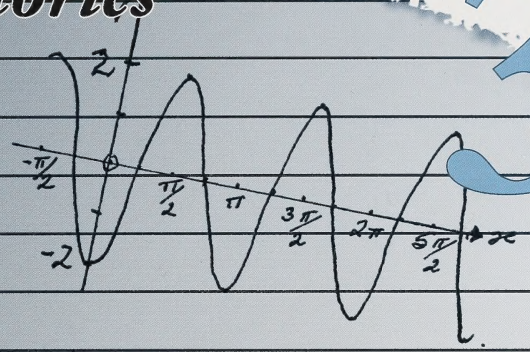
Module 6

MATHEMATICS 30

Combinatorics



$y = 2\sin 2(x - \pi/4)$

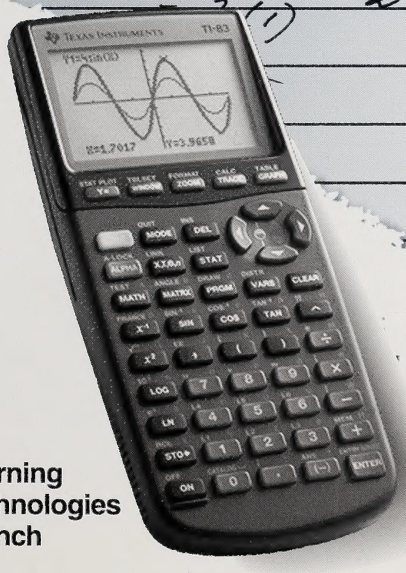
When $x = \pi/2$,

$y = 2\sin 2(\pi/2 - \pi/4)$

$= 2\sin 2(\pi/4)$

$= 2\sin \pi/2$

$= 2(1)$



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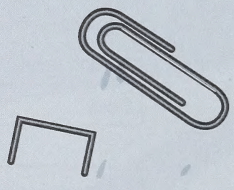
Pure MATHEMATICS 30

Module 6
Combinatorics



Pure Mathematics 30
Module 6: Combinatorics
Student Module Booklet
Learning Technologies Branch
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The Learning Technologies Branch acknowledges with appreciation the Alberta Distance Learning Centre and Pembina Hills Regional Division No. 7 for their review of this Student Module Booklet.

This document is intended for	
Students	✓
Teachers	✓
Administrators	
Home Instructors	
General Public	
Other	

You may find the following Internet sites useful:

- Alberta Education, **http://www.education.gov.ab.ca**
- Learning Technologies Branch, **http://www.education.gov.ab.ca/ltb**
- Learning Resources Centre, **http://www.lrc.education.gov.ab.ca**

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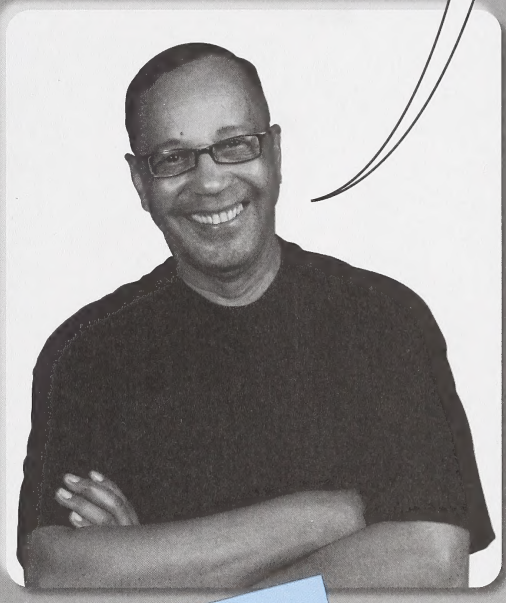
Welcome

Pure Mathematics 30

- **Module 1**
  Transformations
- **Module 2**
  Exponents and Logarithms
- **Module 3**
  Sequences and Series
- **Module 4**
  Conics
- **Module 5**
  Trigonometry
- **Module 6**
  Combinatorics
- **Module 7**
  Probability and Statistics

Welcome to **Module 6**.
We hope you'll enjoy your study of **Combinatorics**.



Module 1 contains general information about the course components, additional resources, icons, assessment, and strategies for completing your work. If you do not have access to Module 1, contact your teacher to obtain this important information.

It is recommended that you work through the modules in the order given (from 1 to 7), since several concepts build on each other as you progress through the course.

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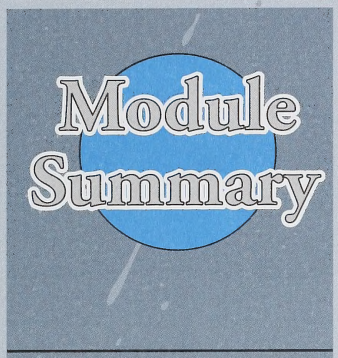

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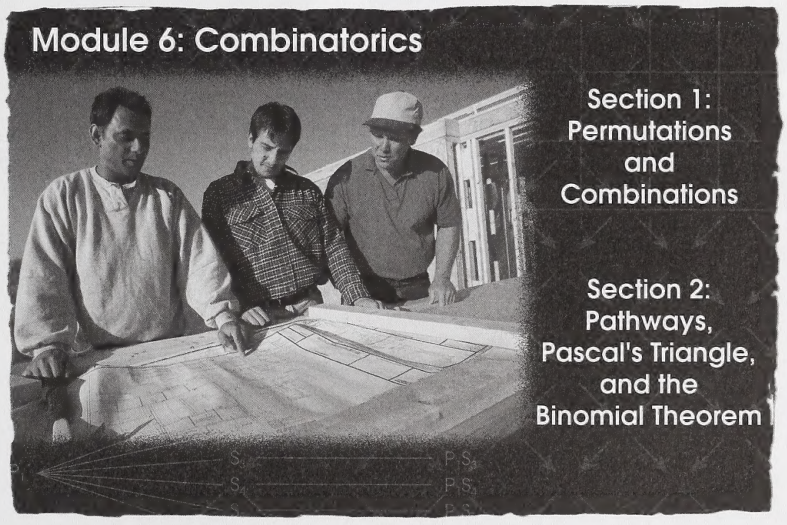
Module Overview

If you have ever participated in a building or renovation project, you know that its success depends on careful planning and organization. Between the initial drawings and the final touches, numerous jobs need to be done, some of which must be performed in a particular order. For example, the foundation and basement must be poured before framing can begin. Other tasks, like electrical work and plumbing, can be done in either order, and, thus, be scheduled at your convenience or when the skilled tradespeople are available.

Itemized lists, flow charts, and diagrams can help you plan and avoid costly mistakes in building and renovation projects. These strategies are also useful in a branch of mathematics called combinatorics. You can use combinatorics to detemine the number of ways tasks can be done.

In this module you will explore the number of ways a combination of tasks can be performed. In your exploration you will make lists, draw diagrams, and derive and apply formulas. In Section 1 you will focus on the number of ways objects or tasks can be arranged and combined. In Section 2 you will build on these skills to solve pathway problems involving Pascal's triangle and the Binomial Theorem.


Module 6: Combinatorics
Section 1: Permutations and Combinations
Section 2: Pathways, Pascal's Triangle, and the Binomial Theorem

Assessment

The document you are presently reading is called a Student Module Booklet. It will show you, step by step, what to do and how to do it.

This module, Combinatorics, has several sections. Within each section, your work is grouped into activities. Within the activities, there are readings and questions for you to answer. By completing these questions you will construct your own learning, discover mathematical connections, and practise or apply what you have learned. The suggested answers in the Appendix of this Student Module Booklet will provide you with immediate feedback on your progress.

In this module you will be directed to the accompanying Assignment Booklets. You are expected to complete two Assignment Booklets for this module. Your grading in this module is based upon the assignments that you submit for assessment. The mark distribution is as follows:

Assignment Booklet 6A	
Section 1 Assignment	**35 marks**
Assignment Booklet 6B	
Section 2 Assignment	**31 marks**
Final Module Assignment	**34 marks**
TOTAL	**100 marks**

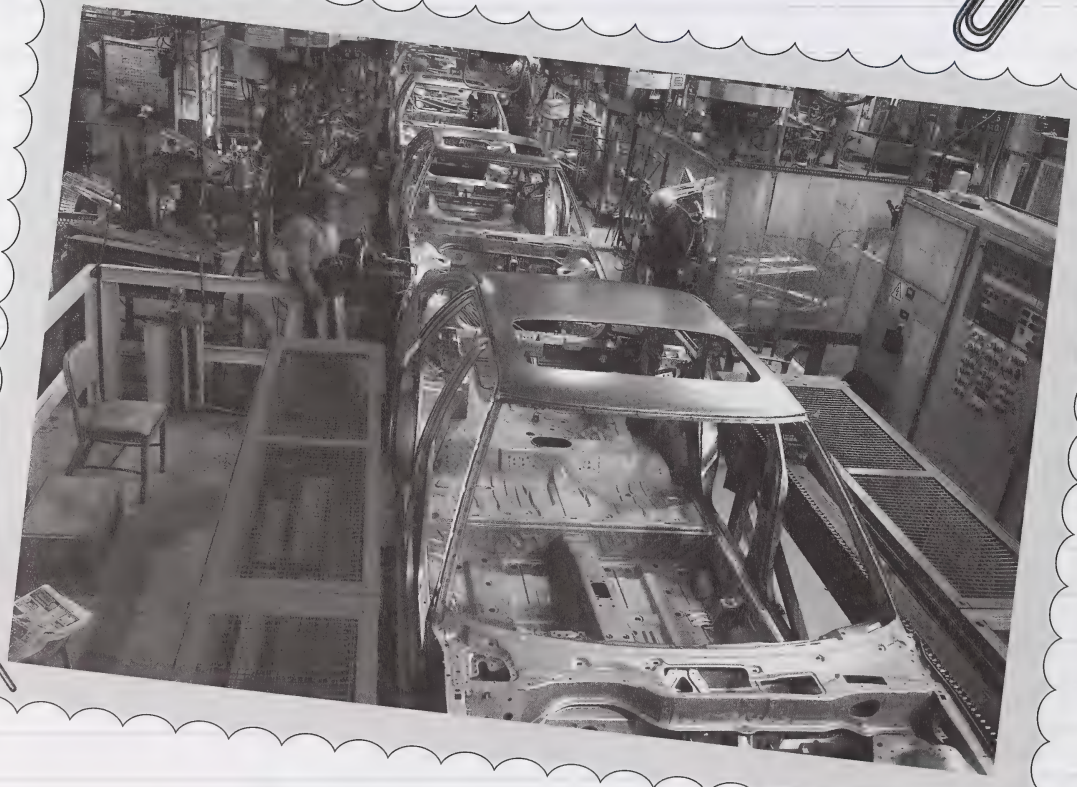
Strategies for Completing a Module

Before you begin, organize your materials: Student Module Booklet, textbook, Assignment Booklets, binder, lined paper, graph paper, graphing calculator, pens, pencils, and so on. Make sure you have a quiet area in which to work, away from distractions. Set up your time schedule.

To achieve success in this module, be sure to read all your instructions carefully and work slowly and systematically through the material. Remember, it's the work you do in this Student Module Booklet that will prepare you for your assignments and final test. Try to set realistic goals for yourself each day; and once you've set them, stick to them. Submit your assignments regularly, and don't forget to review your work before handing it in. Careful work habits will greatly increase your chances for success in Pure Mathematics 30.

Good luck!

Section 1



Permutations and Combinations

Assembly lines are used to manufacture a variety of items, from toasters to vehicles. Before an assembly line can be set up, the company must explore numerous possibilities in order for the assembly line to operate as effectively and efficiently as possible. There are steps in an assembly line that must come before other steps; there are steps that must be done in succession; and there are steps that can appear at many points in the assembly line.

With this in mind, can you imagine the number of possible ways an assembly line of complex items, like cars and trucks, can be set up? To give you an idea, if there are as few as 10 optional steps within an assembly line, then there are as many as 3 628 800 different ways of setting up that assembly line!

In this section you will explore how items can be arranged and combined. This branch of mathematics, called combinatorics, includes the study of permutations and combinations. You will use permutations to investigate the number of ways items can be ordered, and you will use combinations to explore the number of ways you can select an item. Some problem situations will involve both permutations and combinations.

Activity 1: The Fundamental Counting Principle

Nowadays, whenever you call a company for information, especially large corporations, you have to go through a series of recorded messages. For example, you may hear a welcoming statement and then, "For service in English, press 1. For service in French, press 2." Once you have pressed either 1 or 2, you then move to a main menu, where you press a number on your touch-tone phone to get a specific department. Within each main-menu item, a sub-menu may be present. This menu most likely allows you to talk to a specific person or position within the department.

Systems like this are implemented to serve the customer's needs quickly, reducing waiting time and, thus, reducing frustration. This is why the design of these systems is critical.

In order to set up a messaging system, the company needs to know the nature of most customer calls. This way, the majority of the customers should be able to make quick and appropriate choices when they phone. Using diagrams, such as **tree diagrams**, to show how calls will be routed may help select the most efficient design.

In this activity you, too, will use tree diagrams to explore the number of ways several related tasks can be performed. You will then summarize the results of this exploration using the **Fundamental Counting Principle**.

Turn to page 334 of *MATHPOWER 12* and read "The Fundamental Counting Principle."

1. Answer the following on pages 334 and 335 of the textbook.

a. "Explore: Draw a Tree Diagram"
b. questions 1 to 4 of "Inquire"

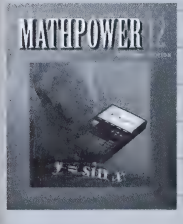
Turn to page 83 to compare your responses with those in the Appendix.

Mathematical Process

- ☐ Communication
- ☒ Connection
- ☐ Estimation
- ☐ Mental Math
- ☐ Problem Solving
- ☐ Reasoning
- ☐ Technology
- ☐ Visualization

tree diagram: a diagram used to illustrate the number of ways several related tasks can be performed

Fundamental Counting Principle: the principle for determining the number of ways two or more operations can be performed together



Mathematical Process
- Communication
- [x] Connection
- Estimation
- Mental Math
- Problem Solving
- Reasoning
- Technology
- Visualization

A tree diagram models the number of ways of performing a multi-stage task. A tree diagram is a very good visual aid; but as the number of ways of doing the stages increases, the tree diagram becomes very tedious to draw. An alternative to drawing a tree diagram is using the Fundamental Counting Principle.

Fundamental Counting Principle

If a task is made up of stages, the total number of ways of performing the task is $m \times n \times p \times \ldots$, where m is the number of ways of performing the first stage, n is the number of ways of performing the second stage, p is the number of ways of performing the third stage, and so on.

The following example uses both methods: a tree diagram and the Fundamental Counting Principle.

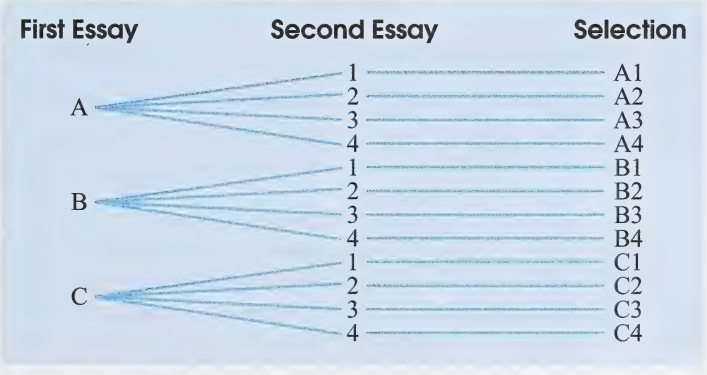
Example

On an English test, a student must write two essays. For the first essay, the student must select from topics A, B, and C. For the second essay, the student must select from topics 1, 2, 3, and 4. How many different ways can the student select the two essay topics?

Solution

Method 1: Using a Tree Diagram

Mathematical Process
- Communication
- Connection
- Estimation
- Mental Math
- Problem Solving
- Reasoning
- Technology
- [x] Visualization



The student can select the two essays 12 ways.

Method 2: Using the Fundamental Counting Principle

The task of choosing essay topics has two stages: choosing an essay from topics A, B, and C and choosing an essay from topics 1, 2, 3, and 4.

The total number of ways of selecting the two essays is the product of the choices available at each of the two stages.

$\therefore$ Number of ways $= 3 \times 4$
$= 12$

The student can select the two essays 12 ways.

By comparing the two methods in the preceding example, note that the Fundamental Counting Principle simply describes how the tree diagram branches outward to all possibilities.

Turn to page 335 of *MATHPOWER 12* and work through Example 1.

2. Answer questions 1 and 3 of "Practice" on page 336 of the textbook.

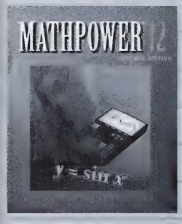
Turn to page 87 to compare your responses with those in the Appendix.

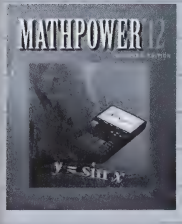
You can use the Fundamental Counting Principle to determine the number of ways of performing a multi-stage task without having to draw a tree diagram that is too cumbersome or time consuming. The following example and questions will provide you with further practice in applying the Fundamental Counting Principle.

Turn to page 336 of *MATHPOWER 12* and work through Example 2.

3. Answer questions 5, 6, and 9 to 14 of "Applications and Problem Solving" on pages 336 and 337 of the textbook.

Turn to page 88 to compare your responses with those in the Appendix.

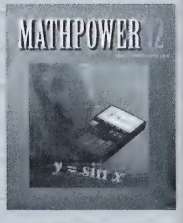




Mathematical Process
- Communication
- Connection
- Estimation
- Mental Math
- [x] Problem Solving
- [x] Reasoning
- Technology
- [x] Visualization

Now Try This

In this activity you saw how a tree diagram aids in visualizing the number of ways a task can be done by listing all of the possibilities. You can also use diagrams along with your powers of deductive reasoning to help you eliminate possibilities. Try using a diagram to help you solve the following puzzle.



4. Turn to page 337 of *MATHPOWER 12* and answer "LOGIC POWER."

Turn to page 91 to compare your response with the one in the Appendix.

Mathematical Process
- [x] Communication
- Connection
- Estimation
- Mental Math
- Problem Solving
- Reasoning
- Technology
- Visualization

Looking Back

In this activity you explored the number of ways multi-stage tasks can be performed using tree diagrams and the Fundamental Counting Principle.

You can use these approaches to analyze your own daily activities. In your journal, make a list of the things you have to do tomorrow morning, afternoon, and evening. How many different ways could you schedule your day?

Activity 2: Permutations

Often, people perform many routine, daily tasks in the same way and in the same order. Think of how you prepare for work or school in the morning. When your day involves a change in routine, then planning is a must to fit everything into a busy schedule so you can accomplish all you want to accomplish.

If you think your life is complicated, think of how complicated the lives of the astronauts on the Apollo 11 moon mission were. They had to perform complex tasks continually on their mission. As a matter of fact, NASA's schedules for the Apollo astronauts were evaluated by a mathematician, Dr. Ronald L. Graham, to ensure that everything could be done in the most efficient manner possible.

If you have access to the Internet, you can find out more about the Apollo moon missions at the following website:

http://www.hq.nasa.gov/office/pao/History/SP-4214/contents.html

In this activity you will explore **permutations**. Setting up a schedule of your school classes or arranging your daily activities are practical examples of permutations. You will shortly discover that the formulas for permutations are based on the Fundamental Counting Principle.

Turn to page 338 of *MATHPOWER 12* and read "Permutations."

1. Answer the following on page 338 of the textbook.

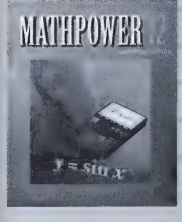
a. "Explore: Look for a Pattern" **b.** questions 1 to 6 of "Inquire"

Turn to page 93 to compare your responses with those in the Appendix.

Mathematical Process

- Communication
- [X] Connection
- Estimation
- Mental Math
- Problem Solving
- Reasoning
- Technology
- Visualization

permutation: an arrangement of all or part of a set of objects, where the order of the arrangement is important



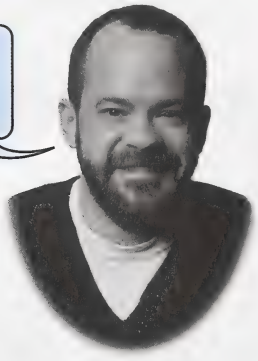
As you have seen from the previous questions, the number of arrangements or permutations of n different objects is as follows:

$$n \times (n-1) \times (n-2) \times (n-3) \times \ldots \times 3 \times 2 \times 1$$

factorial notation: the notation, n!, used to represent the product of the first n natural numbers

In mathematics, there is a notation for a product of this type, called **factorial notation**. It is represented by $n!$ and is read as "n factorial." For example,

$1! = 1$
$2! = 2 \times 1 = 2$
$3! = 3 \times 2 \times 1 = 6$
$4! = 4 \times 3 \times 2 \times 1 = 24$
$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$



By definition, $0! = 1$. The reason for this definition will become apparent as you work your way through the formulas in this module.

Another thing you must remember is that $n!$ is only defined if n is a whole number. This means numbers like 1.5! and (–2)! are undefined. Many calculators, however, will return a value for 1.5! (along with other halves, like 2.5! and 10.5!). The calculator, in this case, is giving a value of the gamma function. The gamma function is a generalization of the factorial for non-natural numbers. For more information, visit the following website:

http://www.rskey.org/gamma.htm

Mathematical Process
- Communication
- Connection
- Estimation
- Mental Math
- Problem Solving
- Reasoning
- [x] Technology
- Visualization

Example

Evaluate each expression using the factorial feature on your graphing calculator.

a. 8! **b.** $\frac{8!}{6!}$

Solution

a. (8) (MATH)
[Select the PRB menu.]
(4) (4:!) (ENTER)

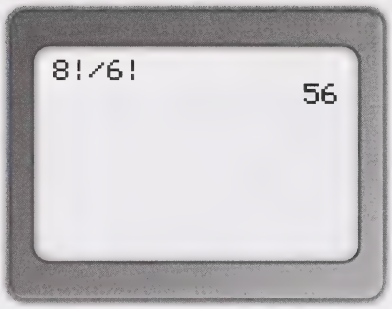
b. 8 MATH

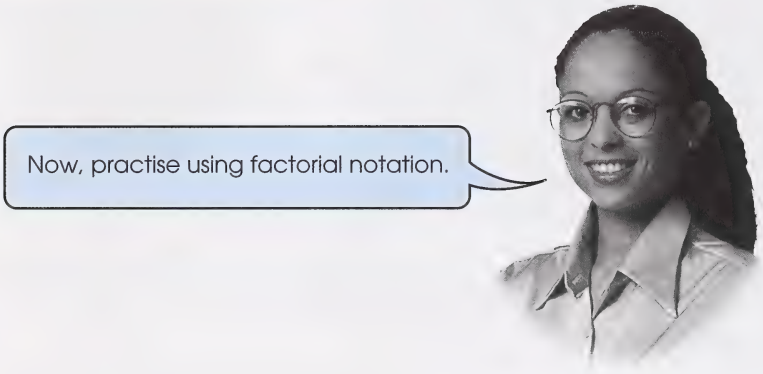
[Select the PRB menu.]

4 (4:!) ENTER ÷ 6

MATH [Select the PRB menu.]

4 (4:!) ENTER





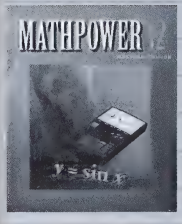
Turn to page 339 of *MATHPOWER 12* and work through Examples 1 and 2.

2. Answer questions 1, 3, 5, 11, 12, and 13 of "Practice" on page 342 of the textbook.

Turn to page 94 to compare your responses with those in the Appendix.

Now that you are familiar with factorial notation, you will investigate its role in formulas for permutations. You will also explore the relationship between these formulas and the Fundamental Counting Principle. Recall from the first part of this activity that the number of permutations of n distinct objects is $n!$. The formula is often written as follows:

$$_nP_n = n!$$



Mathematical Process

- ☐ Communication
- ☒ Connection
- ☐ Estimation
- ☐ Mental Math
- ☐ Problem Solving
- ☐ Reasoning
- ☐ Technology
- ☐ Visualization

The symbol ${}_nP_n$ may be read as "n permute n." It indicates that all n objects are included in each arrangement.

Example

How many different ways can 6 different books be arranged on a shelf?

Solution

Method 1: Using the Formula for ${}_nP_n$

Because all six books are being arranged, $n = 6$.

$$\therefore \ {}_nP_n = n!$$
$${}_6P_6 = 6!$$
$$= 720$$

Six books can be arranged 720 different ways.

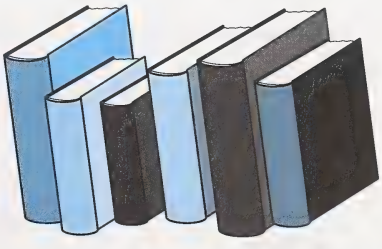
Method 2: Using the Fundamental Counting Principle

Because there are six books to arrange on the shelf, there are six stages in the problem. You must decide what the first book will be, what the second book will be, what the third book will be, and so on.

There are 6 choices for the first book, 5 choices for the second book, 4 choices for the third book, and so on.

$$\therefore \text{ Number of arrangements} = 6 \times 5 \times 4 \times 3 \times 2 \times 1$$
$$= 6!$$
$$= 720$$

Six books can be arranged 720 different ways.



If all permutation questions were this simple, tasks in this activity would be relatively simple. However, in most questions restrictions or conditions are imposed on the arrangements.

Mathematical Process

- Communication
- Connection
- Estimation
- Mental Math
- Problem Solving
- [x] Reasoning
- Technology
- Visualization

Example

Alexa has 3 different math books and 4 different geography books. Determine the number of ways all 7 books can be arranged on the shelf if

a. there are no restrictions
b. the math books are on the left and the geography books are on the right
c. the books in the same subject must be kept together

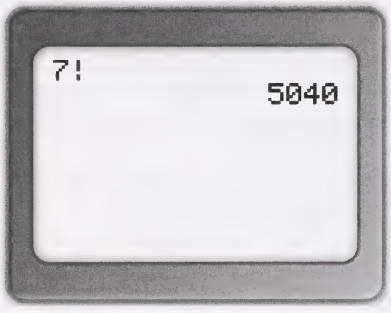
Solution

a. Use the formula for ${}_nP_n$.

Because all 7 books are being arranged, $n = 7$.

$$\therefore {}_nP_n = n!$$
$${}_7P_7 = 7!$$
$$= 5040$$

All seven books can be arranged 5040 ways.



b. There are two stages in this problem. You must arrange the math books first, then arrange the geography books on the right.

The number of ways of arranging the math books is 3!.
The number of ways of arranging the geography books is 4!.

Using the Fundamental Counting Principle,

$$\text{Number of arrangements} = 3! \times 4!$$
$$= 6 \times 24$$
$$= 144$$

There are 144 ways to arrange the math books followed by the geography books.

c. This problem is the same as question b., except that either the math books or the geography books can be on the left.

Therefore, you must multiply the answer to question b. by 2!.

$$\begin{aligned}\text{Number of arrangements} &= 3! \times 4! \times 2! \\ &= 6 \times 24 \times 2 \\ &= 288\end{aligned}$$

There are 288 ways to arrange the seven books so the subjects are kept together.

Mathematical Process
- ☐ Communication
- ☐ Connection
- ☐ Estimation
- ☐ Mental Math
- ☒ Problem Solving
- ☒ Reasoning
- ☐ Technology
- ☐ Visualization

Example

A father, mother, 2 boys, and 3 girls are asked to line up for a photograph. Determine the number of ways they can line up if

a. there are no restrictions
b. the parents stand together
c. the parents do not stand together
d. all the females stand together

Solution

a. Use the formula for ${}_nP_n$.

Since all 7 people must line up, $n = 7$.

$$\begin{aligned}\therefore\ {}_nP_n &= n! \\ {}_7P_7 &= 7! \\ &= 5040\end{aligned}$$

There are 5040 ways the family can line up for a photograph.

b. If the parents stand together, they can be considered a unit, reducing the value of n from 7 to 6.

If $n = 6$, there are 6! arrangements.

However, in each of these arrangements, the parents can switch places and still stand side by side.

$$\therefore \text{ Number of arrangements} = 6! \times 2!$$
$$= 720 \times 2$$
$$= 1440$$

There are 1440 ways the family can line up for the photograph if the parents stand together.

c. Approach this question indirectly.

Recall from the answer to question a. that there were 5040 arrangements possible without any restrictions.

From the answer to question b., there were 1440 arrangements if the parents stood together.

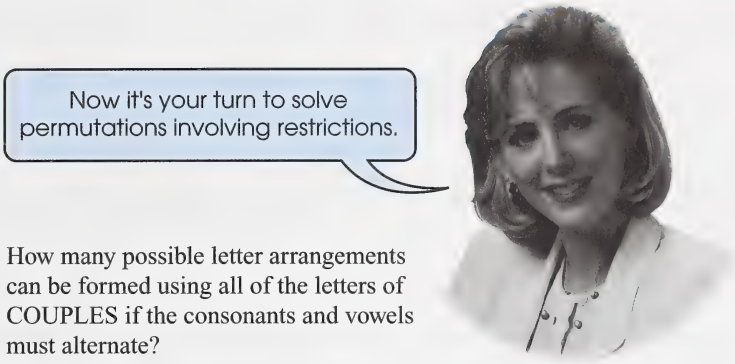
Therefore, if the parents don't stand together, the number of arrangements is

$$\text{Total with no restrictions} - \text{Total with parents together} = 7! - 6! \times 2!$$
$$= 5040 - 1440$$
$$= 3600$$

d. This question is similar to question b. Consider the females as a unit. There are 3 other people in addition to the females. Therefore, $n = 4$. When, $n = 4$, there are 4! possible arrangements. But, even though the 4 females are standing together, they can be arranged among themselves in 4! ways.

$$\therefore \text{ Number of arrangements} = 4! \times 4!$$
$$= 24 \times 24$$
$$= 576$$

There are 576 different ways the family can line up if the females stand together.



3. How many possible letter arrangements can be formed using all of the letters of COUPLES if the consonants and vowels must alternate?

4. How many even, 4-digit numbers can be formed using 2, 3, 4, and 5?

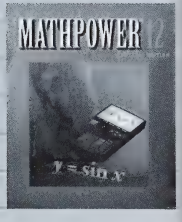
5. How many different ways can 4 math books and 5 other books be arranged on a shelf if the math books are kept together?

6. How many arrangements are possible of all of the letters of CANTER if the vowels cannot be adjacent?

7. Aaron has 3 math books, 4 science books, and 5 history books. How many different ways can these books be arranged on a shelf if books of the same subject must be kept together?

Turn to page 95 to compare your responses with those in the Appendix.

In the preceding questions the objects being arranged were all different. What if some of the objects were identical? How many different arrangements would there be?



Turn to page 339 of *MATHPOWER 12* and read from the red line near the bottom of the page to the red line on page 341, working through Example 3.

When doing the following questions, remember to apply the following principle:

Principle

The number of distinguishable permutations of n *objects, of which* a *objects are identical, another* b *objects are identical, and another* c *objects are identical, and so on, is*

$$\frac{n!}{a!b!c!\ldots}$$

8. Determine the number of distinguishable arrangements for each of the following words.

a. SASKATOON
b. MISSISSIPPI

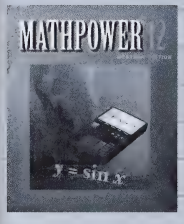
9. How many different numbers can be formed by using all of the following digits?

2, 2, 4, 4, and 5

Turn to page 97 to compare your responses with those in the Appendix.

In the last part of this activity you will investigate situations in which n objects are available, but not all of them are used to form arrangements. For example, suppose you have 10 books and only 3 spaces for them on a shelf. How many ways could you fill those spaces on the shelf? To discover how to answer this type of problem, read on!

Turn to pages 341 of *MATHPOWER 12* and read from the red line to the bottom of the page, working through Example 4.



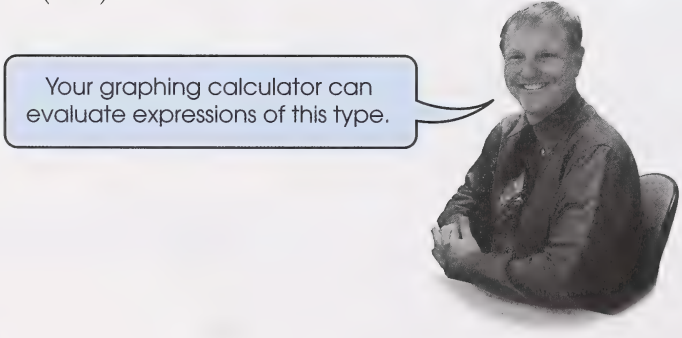
10. Answer the following on page 342 of the textbook.

a. questions 8, 9, 10, 16, 18, and 20 of "Practice"
b. questions 21 to 25.a. of "Applications and Problem Solving"

Turn to page 97 to compare your responses with those in the Appendix.

As you discovered, the formula for the number of permutations of n objects taken r at a time is as follows:

$$ {}_nP_r = \frac{n!}{(n-r)!}, \text{ where } r \leq n $$



Example

Evaluate ${}_{10}P_3$ using pencil and paper and a graphing calculator.

Solution

Method 1: Using Pencil and Paper

$$
\begin{aligned}
{}_{10}P_3 &= \frac{10!}{(10-3)!} \\
&= \frac{10!}{7!} \\
&= 10 \times 9 \times 8 \\
&= 720
\end{aligned}
$$

Mathematical Process
- Communication
- Connection
- Estimation
- Mental Math
- Problem Solving
- Reasoning
- [x] Technology
- Visualization

Method 2: Using a Graphing Calculator

1 0 MATH [Select the PRB menu.]

2 (2:nPR) 3 ENTER

The formula ${}_nP_r = \frac{n!}{(n-r)!}$, where $r \le n$, is consistent with the formula ${}_nP_n = n!$.

Mathematical Process
- Communication
- [x] Connection
- Estimation
- Mental Math
- Problem Solving
- Reasoning
- Technology
- Visualization

If $r = n$, then

$$
\begin{aligned}
{}_nP_r &= \frac{n!}{(n-r)!} \\
{}_nP_n &= \frac{n!}{(n-n)!} \\
&= \frac{n!}{0!} \\
&= \frac{n!}{1} \\
&= n!
\end{aligned}
$$

As you have just seen, one of the reasons why 0! must be defined as 1 is so that the formulas for ${}_nP_r$ and ${}_nP_n$ are consistent. In the next example, you will see how the formula for ${}_nP_r$ can be used in an equation.

Example

Solve $2\left({}_nP_2\right)=60$ for n.

Solution

$$2\left({}_nP_2\right)=60$$

$${}_nP_2=30$$

$$\frac{n!}{(n-2)!}=30 \quad \leftarrow \text{Apply the formula for } {}_nP_r.$$

$$\frac{n(n-1)\cancel{(n-2)!}}{\cancel{(n-2)!}}=30$$

$$n(n-1)=30$$

$$n^2-n-30=0$$

$$(n-6)(n+5)=0$$

$$\therefore\ n-6=0 \quad \text{or} \quad n+5=0$$

$$n=6 \qquad\qquad n=-5$$

However, for ${}_nP_2$ to be defined, the value of n must be a whole number and $n \geq 2$.

Therefore, $n=6$.

11. Solve each of the following for n.

a. ${}_nP_2 = 72$

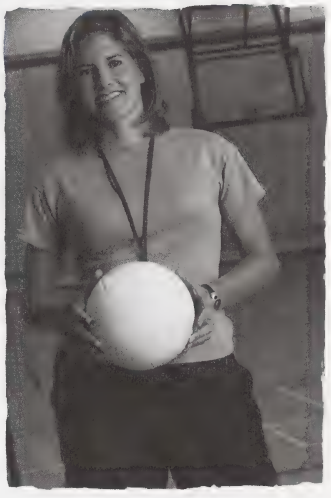
b. ${}_nP_3 = 42n$

Turn to page 101 to compare your responses with those in the Appendix.

Mathematical Process

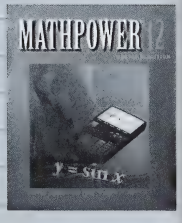
- ☐ Communication
- ☒ Connection
- ☐ Estimation
- ☐ Mental Math
- ☒ Problem Solving
- ☐ Reasoning
- ☐ Technology
- ☐ Visualization

Now Try This



All of the questions and examples that you have done have involved arranging objects in a particular order. In the next activity, which deals with combinations, order will not be relevant. The first step after reading any combinatoric problem is to decide whether or not order is important. For example, coaching staffs of sports teams, especially professional sports teams, are always concerned with wins, losses, and ties and the order in which they occur.

12. Give one reason why the coach of a team would be concerned about the order in which the wins, losses, and ties occur.



13. Turn to page 342 of *MATHPOWER 12* and answer question 27 of "Applications and Problem Solving."

Turn to page 102 to compare your responses with those in the Appendix.

Mathematical Process

- ☒ Communication
- ☐ Connection
- ☐ Estimation
- ☐ Mental Math
- ☐ Problem Solving
- ☐ Reasoning
- ☐ Technology
- ☐ Visualization

Looking Back

In this activity you explored permutations of n objects, permutations of n objects taken r at a time, permutations with restrictions, and distinguishable permutations.

In your journal, distinguish among these types of permutations. For each type of permutation, give at least one example to illustrate the procedures for solving related problems. Discuss what you have written with another student taking Pure Mathematics 30.

Activity 3: Combinations

If you have gone shopping for clothes with friends or family, you have probably had to wait patiently until they made a decision as to what to purchase. To help customers make up their minds, salesclerks lay out a couple of pairs of pants and several shirts. This way, customers can quickly make comparisons and select the preferred combination.

In mathematics, a selection from a set of objects is also called a **combination**. When a selection is made, the order in which the elements are chosen is not important. The same would be true, for example, when choosing a shirt and pants combination. It doesn't matter which item you buy first; what is important is the final look of the outfit.

In this activity you will derive and apply the formula for determining the number of ways a selection from a set can be made when the order does not matter.

Mathematical Process

- Communication
- [x] Connection
- Estimation
- Mental Math
- Problem Solving
- Reasoning
- Technology
- Visualization

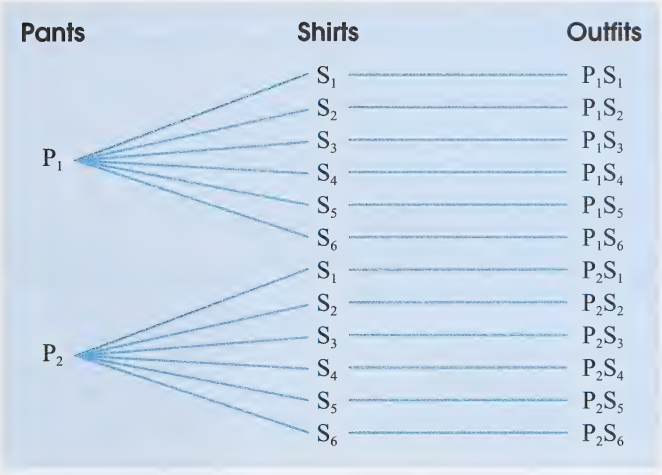
combination: a selection of all or part of a set of objects, where the order of the selection is not important

Example

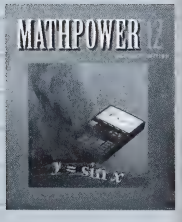
A customer is trying to decide between 2 different pairs of pants and 6 different shirts. How many combinations of 1 pair of pants and 1 shirt are possible?

Solution

Use a tree diagram to illustrate the possible selections. In the diagram, use P_1 and P_2 to represent the pants and S_1, S_2, S_3, S_4, S_5, and S_6 to represent the shirts.



Using the Fundamental Counting Principle, $2 \times 6 = 12$ outfits are possible.



Turn to page 345 of *MATHPOWER 12* and read "Combinations."

1. Answer the following on page 345 of the textbook.

a. "Explore: Look for a Pattern"
b. questions 1 to 4 of "Inquire"

Turn to page 103 to compare your responses with those in the Appendix.

Now, you will work through an example; then you will derive a formula related to the conjecture you just made about the relationship between permutations and combinations.

Example

A tray contains 4 different cookies.

a. How many ways can you select and arrange 3 cookies from the tray?

b. How many ways can you select 3 cookies from the tray if order doesn't matter?

c. How are the answers to questions a. and b. related?

Solution

a. Order is important. Therefore, this is a permutation of 4 objects taken 3 at a time.

$\therefore n = 4$ and $r = 3$

$$\therefore {}_nP_r = {}_4P_3$$
$$= \frac{4!}{(4-3)!}$$
$$= \frac{4!}{1!}$$
$$= 4 \times 3 \times 2$$
$$= 24$$

There are 24 ways of selecting and arranging three cookies from the tray.

b. Let the set $\{A, B, C, D\}$ represent the cookies on the tray.

The three-cookie combinations are $\{A, B, C\}$, $\{A, B, D\}$, $\{A, C, D\}$, and $\{B, C, D\}$.

There are 4 ways of selecting three cookies from the tray if order does not matter.

c. The symbol used to represent the number of combinations is ${}_4C_3$, which is read "4 choose 3." This implies, "From 4, choose 3." Here, ${}_4C_3 = 4$.

The answers to questions a. and b. were 24 and 4 respectively.

Notice that ${}_4P_3 = {}_4C_3 \times 6$ or ${}_4C_3 \times 3!$.

There are 6 or 3! times as many permutations as combinations because there are 3! ways of arranging the three cookies once they are chosen.

The permutation ${}_nP_r$ represents two tasks: selecting r objects from a set of n objects first, then arranging the r objects selected. Using the Fundamental Counting Principle,

$${}_nP_r = \text{number of selections} \times \text{number of arrangements for each solution}$$

$${}_nP_r = {}_nC_r \times r!$$

$${}_nC_r = \frac{{}_nP_r}{r!}$$

$${}_nC_r = \frac{\frac{n!}{(n-r)!}}{r!}, \text{ where } n \geq r$$

$${}_nC_r = \frac{n!}{(n-r)!r!}, \text{ where } n \geq r$$

Note: ${}_nC_r$ may also be written $\binom{n}{r}$

Example

How many different 3-member committees can be selected from a class of 20 students?

Solution

Order is not important; so, determine the number of combinations.

Method 1: Using Pencil and Paper

$n = 20$ and $r = 3$

$$\therefore {}_nC_r = \frac{n!}{(n-r)!r!}$$

$${}_{20}C_3 = \frac{20!}{(20-3)!3!}$$

$$= \frac{20!}{17!3!}$$

$$= \frac{20 \times 19 \times \overset{3}{\cancel{18}} \times \overset{1}{\cancel{17!}}}{\underset{1}{\cancel{17!}} \times \left(3 \times \underset{1}{\cancel{2}} \times 1\right)}$$

$$= 1140$$

There are 1140 different committees that can be selected.

Method 2: Using a Graphing Calculator

(2) (0) (MATH) [Select the PRB menu.]

(3) (3:nCr) (3) (ENTER)

There are 1140 different committees that can be selected.

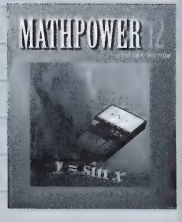
Mathematical Process

- Communication
- Connection
- Estimation
- Mental Math
- Problem Solving
- Reasoning
- [x] Technology
- Visualization

The history of combinatorics and probability is imbedded in games of chance; therefore, many examples and questions on the diploma examination involve a standard deck of cards. If you are not familiar with what a standard deck of cards is made up of, read on. If you are familiar with a standard deck of cards, skip to the textbook icon near the bottom of this page.

A standard deck consists of 52 cards divided into 4 suits: clubs, spades, hearts, and diamonds.

Each suit has 13 cards: 1 ace, 9 numbered cards (from 2 to 10), and 3 face cards (jack, queen, and king).



Turn to pages 346 and 347 of *MATHPOWER 12* and work through Examples 1 and 2.

2. Answer the following on page 348 of the textbook.

a. questions 2, 3, 6, 9, 10, 11, and 14 of "Practice"
b. question 15 of "Applications and Problem Solving"

Turn to page 106 to compare your responses with those in the Appendix.

In the examples that follow, you will use the formula for ${}_nC_r$ to discover additional reasons why 0! = 1.

Example

A cookie jar contains 5 cookies. How many ways can a child choose all 5 cookies if order doesn't matter?

Solution

Because the order doesn't matter, determine the number of combinations.

$n = 5$ and $r = 5$

$$\therefore {}_nC_r = \frac{n!}{(n-r)!r!}$$

$${}_5C_5 = \frac{5!}{(5-5)!\,5!}$$

$$= \frac{1}{0!}$$

$$= 1$$

If order is not important, the child can choose all five cookies only 1 way!

Example

A cookie jar contains 5 cookies. A child is offered a cookie, but refuses. How many ways can this be done?

Solution

$n = 5$ and $r = 0$

$$\therefore {}_nC_r = \frac{n!}{(n-r)!r!}$$

$${}_5C_0 = \frac{5!}{(5-0)!0!}$$

$$= \frac{\cancel{5!}}{\cancel{5!}0!}$$

$$= \frac{1}{0!}$$

$$= 1$$

There is only 1 way the child can reject the cookie. This is done simply by saying, "No."

In the questions and examples you have just completed, there weren't any additional restrictions or conditions imposed. Combination problems with restrictions can be quite challenging. Following are some examples to help you solve these problems.

Example

A Pure Mathematics 30 class has 10 boys and 12 girls. The teacher wants to form a committee of 3 students to plan the year-end picnic. Determine the number of committees possible if

a. there are no restrictions
b. there are no boys on the committee
c. there must be at least one boy on the committee

Solution

a. $n = 22$ and $r = 3$

$$\therefore {}_nC_r = \frac{n!}{(n-r!)r!}$$

$${}_{22}C_3 = \frac{22!}{(22-3)!3!}$$

$$= \frac{22!}{19!3!}$$

$$= \frac{22 \times 21 \times 20 \times 19!}{19! \times 3 \times 2 \times 1}$$

$$= 1540$$

Without restrictions, there are 1540 committees possible.

b. Because there are no boys on the committee, you must select committee members from the 12 girls.

$\therefore n = 12$ and $r = 3$

$$\therefore {}_nC_r = \frac{n!}{(n-r)!r!}$$

$${}_{12}C_3 = \frac{12!}{(12-3)!3!}$$

$$= \frac{12!}{9!3!}$$

$$= \frac{12 \times 11 \times 10 \times 9!}{9!(3 \times 2 \times 1)}$$

$$= 220$$

There are 220 possible committees of only girls.

c. Method 1: Considering All Possible Cases

Case 1: 1 Boy and 2 Girls

Because there are 10 boys and 12 girls to choose from, apply the Fundamental Counting Principle.

$$\begin{aligned}\text{Number of committees} &= {}_{10}C_1 \times {}_{12}C_2 \\ &= \frac{10!}{(10-1)!1!} \times \frac{12!}{(12-2)!2!} \\ &= \frac{10!}{9!} \times \frac{12!}{10!2!} \\ &= \frac{10 \times \overset{1}{\cancel{9!}}}{\underset{1}{\cancel{9!}}} \times \frac{\overset{6}{\cancel{12}} \times 11 \times \overset{1}{\cancel{10!}}}{\underset{1}{\cancel{10!}} \times \underset{1}{\cancel{2}} \times 1} \\ &= 10 \times 66 \\ &= 660\end{aligned}$$

Case 2: 2 Boys and 1 Girl

Applying the Fundamental Counting Principle,

$$\begin{aligned}\text{Number of committees} &= {}_{10}C_2 \times {}_{12}C_1 \\ &= \frac{10!}{(10-2)!2!} \times \frac{12!}{(12-1)!1!} \\ &= \frac{10!}{8!2!} \times \frac{12!}{11!} \\ &= \frac{\overset{5}{\cancel{10}} \times 9 \times \overset{1}{\cancel{8!}}}{\underset{1}{\cancel{8!}} \times \underset{1}{\cancel{2}} \times 1} \times \frac{12 \times \overset{1}{\cancel{11!}}}{\underset{1}{\cancel{11!}}} \\ &= 45 \times 12 \\ &= 540\end{aligned}$$

Case 3: 3 Boys

$$\begin{aligned}\text{Number of committees} &= {}_{10}C_3 \\ &= \frac{10!}{(10-3)!3!} \\ &= \frac{10!}{7!3!} \\ &= \frac{\overset{5}{\cancel{10}} \times \overset{3}{\cancel{9}} \times 8 \times \overset{1}{\cancel{7!}}}{\underset{1}{\cancel{7!}} \times \underset{1}{\cancel{3}} \times \underset{1}{\cancel{2}} \times 1} \\ &= 120\end{aligned}$$

$$\begin{aligned}\therefore \text{ Total number of committees} &= {}_{10}C_1 \times {}_{12}C_2 + {}_{10}C_2 \times {}_{12}C_1 + {}_{10}C_3 \\ &= 660 + 540 + 120 \\ &= 1320\end{aligned}$$

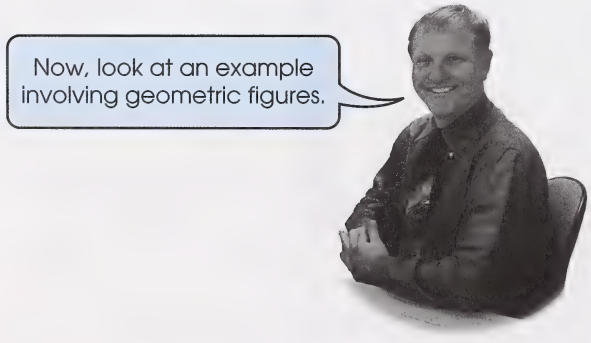
There are 1320 possible committees with at least one boy.

Method 2: Using an Indirect Approach

From the answer to question a., you know there are 1540 committees possible when there are no restrictions. Some of these committees will be all boys, others will be all girls, and the rest will be boys and girls. From the answer to question b., you know there are 220 committees that consist of only girls.

$$\begin{aligned}\therefore \text{ Total number of committees with at least one boy} &= 1540 - 220 \\ &= 1320\end{aligned}$$

There are 1320 possible committees with at least one boy.



Example

How many diagonals are there in a regular octagon?

Solution

A regular octagon has 8 sides and 8 vertices. Choose any two vertices and join them. The line segment formed is either a side or a diagonal. The total number of ways of joining the 8 vertices is ${}_8C_2$.

${}_8C_2$ = number of sides + number of diagonals

$\therefore$ Number of diagonals = ${}_8C_2$ − number of sides

$= 28 - 8$

$= 20$

There are 20 diagonals in a regular octagon.

The number of diagonals of any regular polygon with n sides can be determined using the following equation:

Number of diagonals = ${}_nC_2 - n$

Now, answer the following questions.

3. How many diagonals are in a regular polygon with 20 sides?

4. How many triangles can be formed by joining the vertices of a regular polygon with 20 sides?

Turn to page 109 to compare your responses with those in the Appendix.

Now, you will solve equations involving a combination. Work through the following example.

Example

If $3\left({}_nC_2\right)=30$, solve for n.

Solution

$$3\left({}_nC_2\right)=30$$
$${}_nC_2=10$$
$$\frac{n!}{(n-2)!2!}=10$$
$$\frac{n(n-1)\cancel{(n-2)!}}{\cancel{(n-2)!}\times 2\times 1}=10$$
$$\frac{n(n-1)}{2}=10$$
$$n(n-1)=20$$
$$n^2-n=20$$
$$n^2-n-20=0$$
$$(n-5)(n+4)=0$$

$\therefore\ n-5=0$ or $n+4=0$

$n=5$ $\qquad$ $n=-4$

Because ${}_nC_2$ is only defined when $n\geq 2$, where n is a whole number, $n=5$.

Mathematical Process
- ☐ Communication
- ☒ Connection
- ☐ Estimation
- ☐ Mental Math
- ☒ Problem Solving
- ☒ Reasoning
- ☐ Technology
- ☒ Visualization

Turn to pages 347 and 348 of *MATHPOWER 12* and work through Example 3.

5. Answer questions 16, 17, 18, 19.b., 19.e., 19.f., 20, 21, and 23 of "Applications and Problem Solving" on pages 348 and 349 of the textbook.

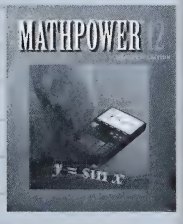
Turn to page 109 to compare your responses with those in the Appendix.

Now Try This

It is often claimed that chess masters are able to think many moves ahead, anticipating their opponents' moves and countering these moves to gain an advantage. As a matter of fact, success at many games and puzzles depends on the ability to think ahead and make logical choices. You will need these skills to solve the next puzzle.

6. Turn to page 349 of *MATHPOWER 12* and answer "NUMBER POWER."

Turn to page 118 to compare your response with the one in the Appendix.



Mathematical Process
- ☒ Communication
- ☐ Connection
- ☐ Estimation
- ☐ Mental Math
- ☐ Problem Solving
- ☐ Reasoning
- ☐ Technology
- ☐ Visualization

Looking Back

In this activity you explored combinations and their related formulas. You discovered that, unlike permutations, order for combinations does not matter. Throughout this activity, you practised solving numerous real-world problems involving combinations.

By now, you have probably used a combination lock, whether for a locker at school, at a local swimming pool, or even for the lock on your bicycle. In your journal, discuss whether "combination lock" is an appropriate name for the lock.

Follow-up Activities

If you had difficulties understanding the concepts and skills in the activities, it is recommended that you do the Extra Help. If you have a clear understanding of the concepts and skills, it is recommended that you do the Enrichment. You may decide to do both.

Extra Help

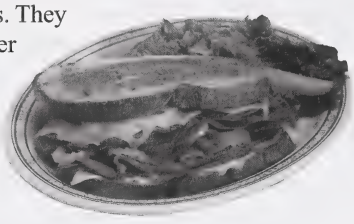
Many students have difficulty deciding whether a question should be answered using a permutation, a combination, or the Fundamental Counting Principle. Often, you are not told directly whether or not order matters. You have to make that decision from the context of the problem. Carefully work through the following examples. Pay close attention to the approaches taken in each case.

Example

Mathematical Process

- Communication
- Connection
- Estimation
- Mental Math
- [x] Problem Solving
- Reasoning
- Technology
- Visualization

A local deli is well known for their sandwiches. They advertise that customers can choose any number of the following toppings for their sandwich: mayonnaise, butter, tomatoes, peppers, lettuce, and pickles. How many different ham sandwiches are possible?



Solution

Method 1: Using the Fundamental Counting Principle

One way to visualize the problem is to think of each topping as an answer to a yes/no question. For each of the six possible toppings, you have two choices: yes you want the topping or no you do not want the topping.

Using the Fundamental Counting Principle, the number of possible sandwiches, n, is

$$\begin{aligned} n &= 2 \times 2 \times 2 \times 2 \times 2 \times 2 \\ &= 2^6 \\ &= 64 \end{aligned}$$

The deli can make 64 different ham sandwiches.

Method 2: Using Combinations

You can select no toppings, one topping, two toppings, and so on. Therefore, the total number of possible sandwiches, n, is found by adding these combinations. Here, the order of the toppings does not matter.

$$\begin{aligned} n &= {}_6C_0 + {}_6C_1 + {}_6C_2 + {}_6C_3 + {}_6C_4 + {}_6C_5 + {}_6C_6 \\ &= 1 + 6 + 15 + 20 + 15 + 6 + 1 \\ &= 64 \end{aligned}$$

The deli can make 64 different ham sandwiches.

Mathematical Process

- Communication
- Connection
- Estimation
- Mental Math
- [x] Problem Solving
- [x] Reasoning
- Technology
- Visualization

Example

To work on the school clean-up committee, only 4 people are needed. If 7 students volunteer to work on the committee,

a. determine the number of 4-member committees possible

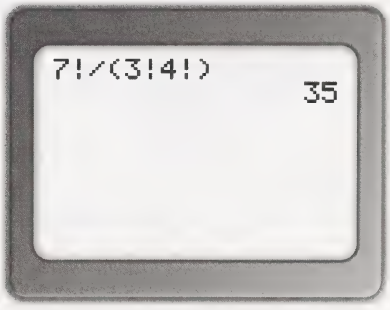
b. determine the number of 4-member committees possible if one member is responsible for collecting garbage, one member is responsible for collecting containers for recycling, and two members are responsible for the grounds

Solution

a. $n = 7$ and $r = 4$

$$\begin{aligned} \therefore {}_nC_r &= \frac{n!}{(n-r)!r!} \\ {}_7C_4 &= \frac{7!}{(7-4)!4!} \\ &= \frac{7!}{3!4!} \\ &= 35 \end{aligned}$$

A total of 35 committees are possible.



b. Let n represent the total number of committees possible.

Method 1: Using Combinations and the Fundamental Counting Principle

First, select the person responsible for collecting the garbage from the 7 volunteers. This can be done in ${}_7C_1$ ways.

Second, select the person responsible for collecting the containers for recycling from the remaining 6 volunteers. This can be done in ${}_6C_1$ ways.

Finally, select the 2 people responsible for the grounds from the remaining 5 volunteers. This can be done in ${}_5C_2$ ways.

Using the Fundamental Counting Principle,

$$\begin{aligned} n &= {}_7C_1 \times {}_6C_1 \times {}_5C_2 \\ &= 7 \times 6 \times 10 \\ &= 420 \end{aligned}$$

A total of 420 committees are possible.

Method 2: Using Permutations and the Fundamental Counting Principle

You can think of the problem as first selecting the four committee members, and then assigning tasks.

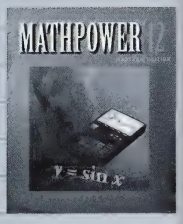
Recall from the answer to question a. that there were ${}_7C_4 = 35$ ways of selecting the four people to serve on the committee.

Once the members are selected, the tasks can then be assigned. The assignment of tasks is like arranging the letters G (garbage), B (bottles), R (grounds), and R (grounds). There are $\frac{4!}{2!} = 12$ distinguishable arrangements of G B R R.

Using the Fundamental Counting Principle,

$$\begin{aligned} n &= {}_7C_4 \times \frac{4!}{2!} \\ &= 35 \times 12 \\ &= 420 \end{aligned}$$

A total of 420 committees are possible.



Next, try some questions on your own.

Turn to page 360 of *MATHPOWER 12* and answer questions 1, 3, 4, 5, 6, and 8 of "Review."

Turn to page 122 to compare your responses with those in the Appendix.

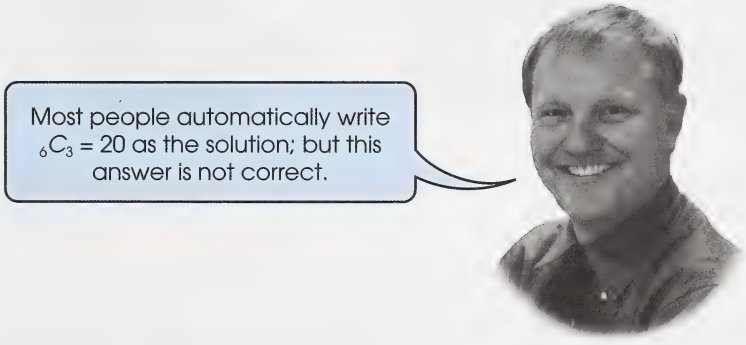
Enrichment

Earlier in this section, you applied the formula for ${}_nC_r$. This formula gave you the number of ways of selecting r objects from a set of n objects. How can you apply this formula to determine the number of ways of dividing the entire set of n objects into subsets, not just selecting a few objects from the total?

Example

In a science lab, 6 students are to be divided into 2 groups of 3 to do an experiment. How many ways can this be done?

Solution



Let A, B, C, D, E, and F represent the students. If you choose ABC for one group, do you agree that the other group is automatically DEF?

Use a table to list all the possibilities.

Group 1	Group 2	Group 1	Group 2
ABC	DEF	ACE	BDF
ABD	CEF	ACF	BDE
ABE	CDF	ADE	BCF
ABF	BDE	ADF	BCE
ACD	BEF	AEF	BCD

There are only 10 ways of dividing six students into two groups of three.

1. Use a table to determine the number of ways 4 people can be divided into 2 groups of 2.

2. Determine the number of ways 8 people can be divided into 2 groups of 4 using a combination.

3. How many ways can $2n$ people be divided into 2 equal groups?

4. Obviously, an odd number of people cannot be divided into two equal groups. However, 6 people and 9 people can be divided into 3 equal groups.

   a. Using a table, show how 6 people (A, B, C, D, E, and F) can be divided into 3 groups of 2.

   b. Determine the number of ways 9 people can be divided into 3 equal groups.

Turn to page 128 to compare your responses with those in the Appendix.

Conclusion

In this section you explored the Fundamental Counting Principle, permutations, and combinations and the relationships among them.

In particular, you examined permutations of entire sets of objects, permutations of subsets, permutations with restrictions, and distinguishable permutations. In each case you derived and applied formulas to solve problems dealing with everyday situations.

In your study of combinations you also examined problems with restrictions. You discovered that order mattered in some problem situations, whereas order was irrelevant in other situations.

The study of combinations and permutations helps answer the question, "How many ways can this task be done?" Being able to answer related questions is important in planning and design. What are the choices in designing an efficient assembly line? How many licence plates consisting of three letters followed by three numbers are possible? How many different telephone area codes are possible? Can you really order 192 different hamburgers at a fast-food outlet?

Assignment

Turn to Assignment Booklet 6A and complete the assignment for Section 1. Remember to submit your completed Assignment Booklet for assessment before starting Section 2.

Section 2

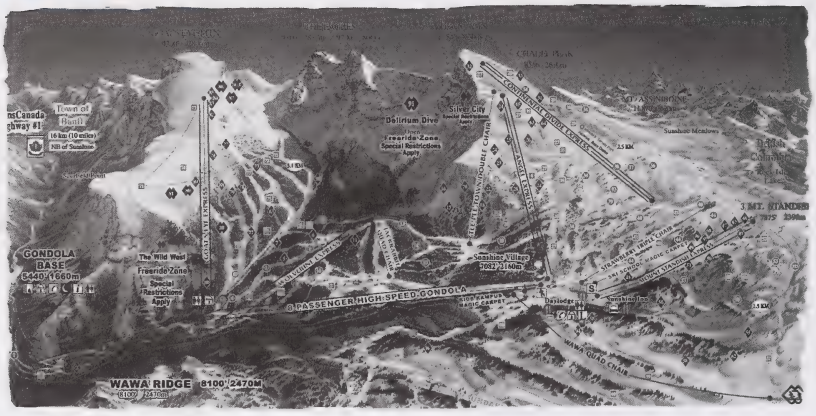
Pathways, Pascal's Triangle, and the Binomial Theorem

Successful people often make their accomplishments look easy. However, when you examine the lives of most successful people, you soon realize that they only accomplished their goals after careful planning and a lot of hard work. Many of these people experienced frustration and adversity along the way. The path to success is seldom straight, and there are many crossroads where important decisions must be made.

Decision making while working your way towards your goals is one type of pathway problem.

In this section you will explore pathway problems. You will discover that a triangular array of numbers, known as Pascal's triangle, can be derived from the analysis of paths in pathway problems. Once you have derived Pascal's triangle, you will examine how you can use it to expand powers of binomials and derive a general procedure for expanding binomials, called the Binomial Theorem.

Activity 1: Pathways and Pascal's Triangle



Sunshine Village Ski Resort

At most ski hills, like Sunshine Village, there are maps that show every possible run that leads to the chalet. Provincial and national parks, too, provide trail maps to people who plan to hike in the back country. Whether it is a ski run, hiking trail, or the path of your education, you are constantly confronted with choices. Informed decisions must be made so you can successfully reach your destination.

In this activity you will investigate the number of ways you can proceed from the beginning of a task to the end of the task. These problems are called **pathway problems**. As part of this investigation, you will derive **Pascal's triangle**, a triangular array of numbers that will assist you in solving pathway problems and also help you expand powers of binomials in the next activity.

You can research the history of Blaise Pascal, the founder of Pascal's triangle, at the following website. You may also enter *Pascal* into any of the Internet's search engines to find more information.

http://www.maths.tcd.ie/pub/HistMath/People/Pascal/RouseBall/RB_Pascal.html

Turn to page 350 of *MATHPOWER 12* and read "Pathways and Pascal's Triangle."

1. Answer the following on pages 350 and 351 of the textbook.

   a. "Explore: Look for a Pattern"
   b. questions 1 to 4 of "Inquire"

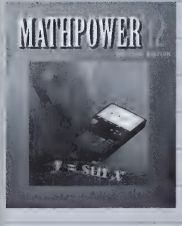
Turn to page 130 to compare your responses with those in the Appendix.

Mathematical Process

- Communication
- [x] Connection
- Estimation
- Mental Math
- Problem Solving
- Reasoning
- Technology
- Visualization

pathway problem: a problem involving the number of ways of going from one point to another

Pascal's triangle: the triangular arrangement of numbers, where 1 is in the first row and each number in the succeeding rows is the sum of the two numbers above it in the preceding row

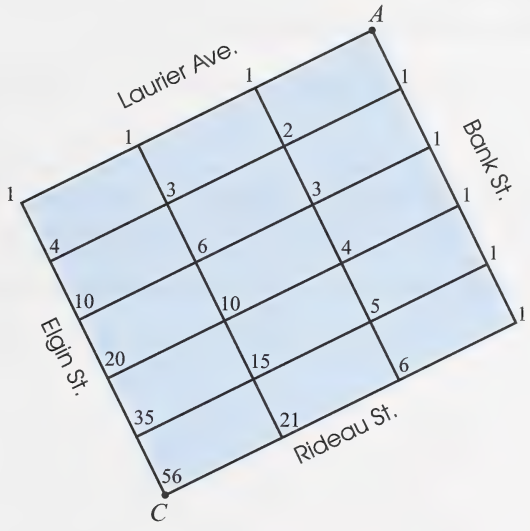


Mathematical Process

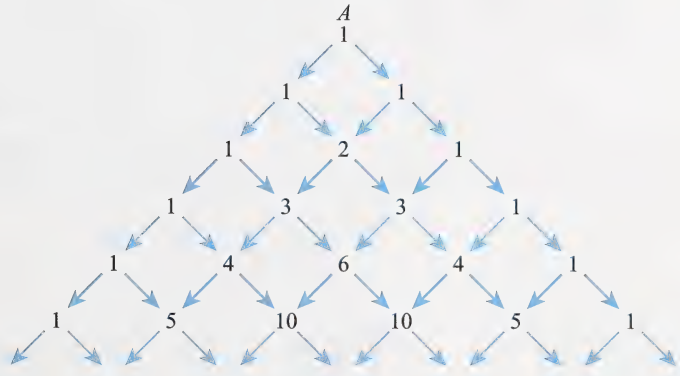
- ■ Communication
- ■ Connection
- ■ Estimation
- ■ Mental Math
- ■ Problem Solving
- ■ Reasoning
- ■ Technology
- ☒ Visualization

In the previous exploration you looked at a simplified Ottawa street map. Starting from Point *A*, you determined the number of ways of reaching each intersection as you traced paths towards Point *C*. You can generalize your results to develop Pascal's triangle.

First, rotate the diagram so that point *A* is at the top. Now, to travel from intersection to intersection, you can either go downward to the right or downward to the left. The number at each intersection indicates the number of ways of reaching that particular intersection.



The numbers at the intersections form Pascal's triangle.



The first and last entry in each row of Pascal's triangle is a 1. Every other entry in Pascal's triangle is the sum of the two entries in the preceding row that lie to the immediate left and right. The following shows how the entries in the last row were determined:

1	5	10	10	5	1
	$(1+4)$	$(4+6)$	$(6+4)$	$(4+1)$	

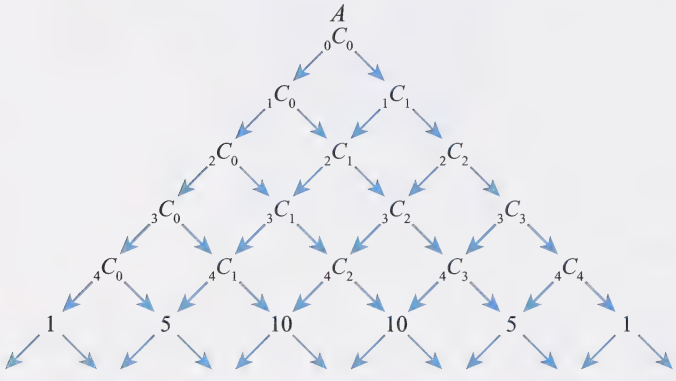
Now, try the following questions.

2. Extend the preceding Pascal's triangle by two rows.

3. Explain why the entries in any row are the same read forwards and backwards.

Turn to page 131 to compare your responses with those in the Appendix.

The entries in Pascal's triangle can also be determined using combinations.



Now, look at the entries in the last row: 1 5 10 10 5 1. Do you agree that the entries can be replaced by ${}_5C_0$ ${}_5C_1$ ${}_5C_2$ ${}_5C_3$ ${}_5C_4$ ${}_5C_5$? From left to right, look at each of the six intersections in this row and determine the number of ways you can reach them.

Case 1: The First Intersection

There is only one way to arrive at this intersection; you must travel 5 blocks to the left (L). This is just like asking the question, "How many distinguishable arrangements are there of L L L L L?"

$$\therefore \text{ Number of arrangements} = \frac{5!}{5!}$$
$$= \frac{5!}{5!0!}$$
$$= \frac{5!}{(5-0)!0!}$$
$$= {}_5C_0 \text{ or } 1$$

Case 2: The Second Intersection

To arrive at this intersection, you must travel 4 blocks to the left (L) and 1 block to the right (R). Two possibilities are L L L L R and L L L R L. How many ways are there altogether? This is just like asking the question, "How many distinguishable arrangements are there of L L L L R?"

$$\therefore \text{ Number of arrangements} = \frac{5!}{4!}$$
$$= \frac{5!}{4!1!}$$
$$= \frac{5!}{(5-1)!1!}$$
$$= {}_5C_1 \text{ or } 5$$

Case 3: The Third Intersection

To arrive at this intersection, you must travel 3 blocks to the left (L) and 2 blocks to the right (R). Two possibilities are L L L R R and L L R R L. How many ways are there altogether? This is just like asking the question, "How many distinguishable arrangements are there of L L L R R?"

$$\therefore \text{ Number of arrangements} = \frac{5!}{3!2!}$$
$$= \frac{5!}{(5-2)!3!}$$
$$= {}_5C_2 \text{ or } 10$$

Case 4: The Fourth Intersection

To arrive at this intersection, you must travel 2 blocks to the left (L) and 3 blocks to the right (R).

$$\therefore \text{ Number of distinguishable arrangements} = \frac{5!}{2!3!}$$
$$= \frac{5!}{(5-3)!3!}$$
$$= {}_5C_3 \text{ or } 10$$

Case 5: The Fifth Intersection

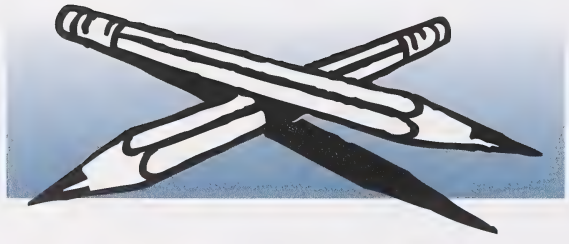
To arrive at this intersection, you must travel 1 block to the left (L) and 4 blocks to the right (R).

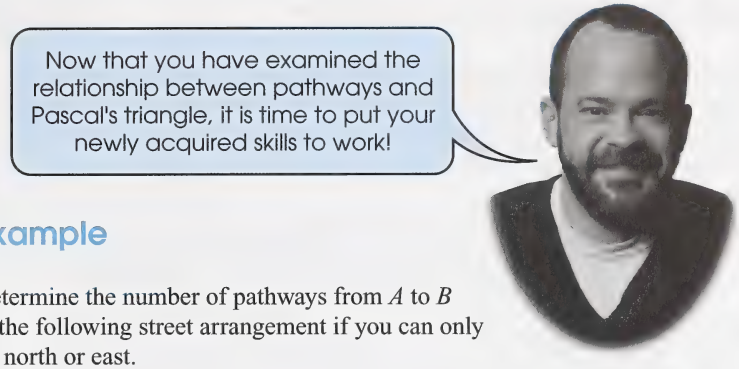
$$\therefore \text{ Number of distinguishable arrangements} = \frac{5!}{1!4!}$$
$$= \frac{5!}{(5-4)!4!}$$
$$= {}_5C_4 \text{ or } 5$$

Case 6: The Sixth Intersection

To arrive at this intersection, you must travel 5 blocks to the right (R).

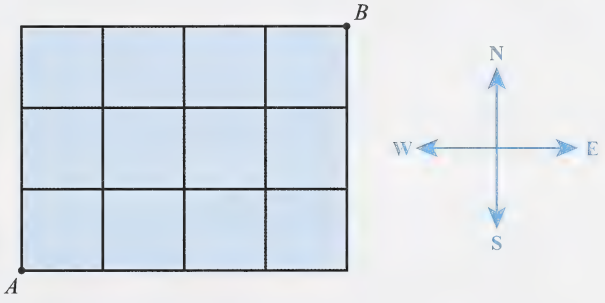
$$\therefore \text{ Number of distinguishable arrangements} = \frac{5!}{0!5!}$$
$$= \frac{5!}{(5-5)!5!}$$
$$= {}_5C_5 \text{ or } 1$$




Now that you have examined the relationship between pathways and Pascal's triangle, it is time to put your newly acquired skills to work!

Example

Determine the number of pathways from A to B in the following street arrangement if you can only go north or east.



Solution

Method 1: Using Entries from Pascal's Triangle

Starting at A and working toward B, insert the entries from Pascal's triangle. Remember, add to obtain missing entries.

There are 35 different paths from A to B.

Mathematical Process

- Communication
- Connection
- Estimation
- Mental Math
- Problem Solving
- Reasoning
- Technology
- ☒ Visualization

Method 2: Using Distinguishable Arrangements

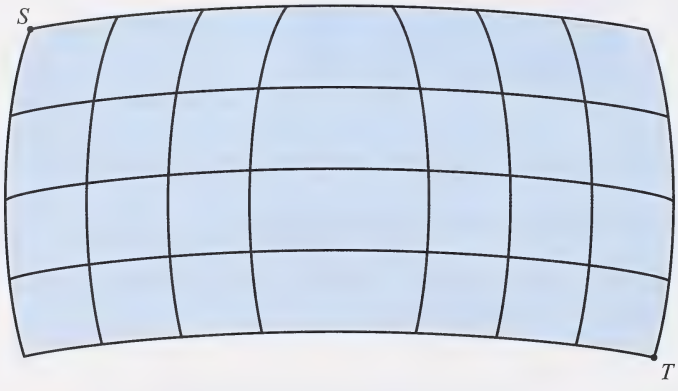
Recall how combinations were used to obtain the rows of Pascal's triangle. To go from A to B, you must travel 4 blocks east (E) and 3 blocks north (N). Therefore, the number of paths from A to B will correspond to the number of distinguishable arrangements of E E E E N N N.

$$\therefore \text{ Number of paths} = \frac{7!}{4!3!}$$
$$= 35$$

There are 35 different paths from A to B.

Example

How many paths are there from S to T if no backtracking is allowed?

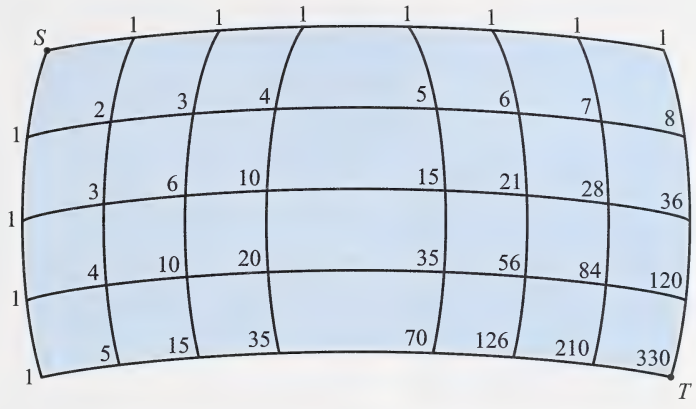


Mathematical Process
- Communication
- Connection
- Estimation
- Mental Math
- Problem Solving
- Reasoning
- Technology
- [X] Visualization

Solution

Method 1: Using Pascal's Triangle

Starting at S and working toward T, apply the techniques of Pascal's triangle.



There are 330 paths from S to T.

Method 2: Using Distinguishable Arrangements

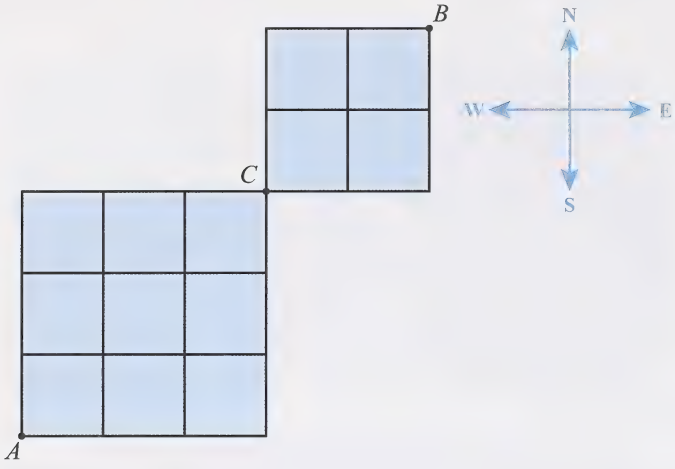
To go from S to T, you need to go 7 blocks right and 4 blocks down. This corresponds to the number of arrangements of R R R R R R R D D D D.

$$\therefore \text{ Number of paths} = \frac{11!}{7!4!}$$
$$= 330$$

There are 330 paths from S to T.

Example

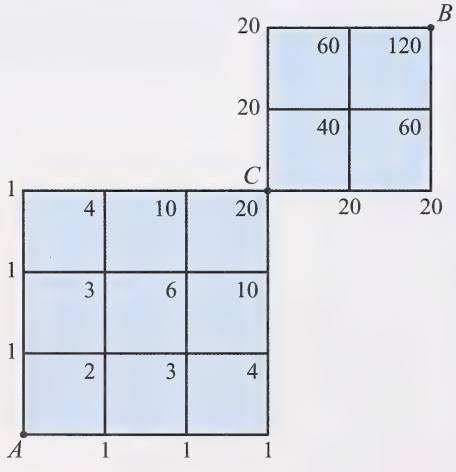
If you can only travel north or east, determine the number of pathways from A to B in the following street arrangement.



Solution

Method 1: Using Pascal's Triangle

Starting at A and working toward B, apply the techniques of Pascal's triangle.



There are 120 different paths from A to B.

Mathematical Process

- Communication
- Connection
- Estimation
- Mental Math
- Problem Solving
- Reasoning
- Technology
- [x] Visualization

Method 2: Using Distinguishable Arrangements and the Fundamental Counting Principle

First, calculate the number of paths from A to C.

To go from A to C, you must travel 3 blocks (E) and 3 blocks north (N). Therefore, the number of paths from A to C will correspond to the number of distinguishable arrangements of E E E N N N.

$$\therefore \text{ Number of paths from } A \text{ to } C = \frac{6!}{3!3!}$$
$$= 20$$

Next, calculate the number of paths from C to B.

To go from C to B, you must travel 2 blocks east (E) and 2 blocks north (N). Therefore, the number of paths from C to B will correspond to the number of distinguishable arrangements of E E N N.

$$\therefore \text{ Number of paths from } C \text{ to } B = \frac{4!}{2!2!}$$
$$= 6$$

Using the Fundamental Counting Principle,

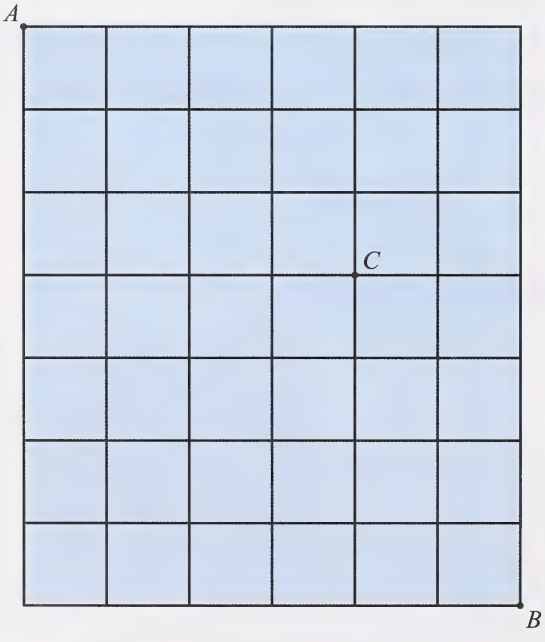
Number of paths from A to B
$= \text{number of paths from } A \text{ to } C \times \text{number of paths from } C \text{ to } B$
$= 20 \times 6$
$= 120$

There are 120 different paths from A to B.

Turn to pages 351 and 352 of *MATHPOWER 12* and work through Examples 1 and 2.

Example

If no backtracking is allowed, how many paths are there from A to B that go through C?

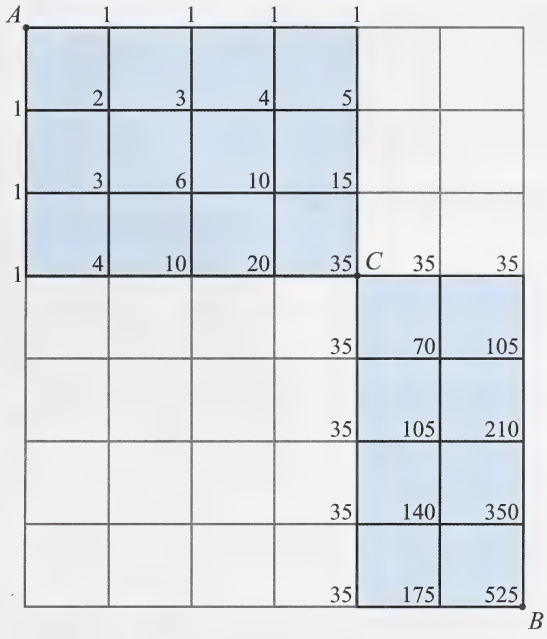


Mathematical Process
- Communication
- Connection
- Estimation
- Mental Math
- Problem Solving
- Reasoning
- Technology
- [x] Visualization

Solution

Method 1: Using Pascal's Triangle

Starting at point A, apply the techniques of Pascal's triangle to reach point C and then on to point B. The grey areas of the diagram cannot be used because those paths would involve backtracking.



There are 525 paths from A to B that pass through C.

Method 2: Using Distinguishable Arrangements and the Fundamental Counting Principle

Break the problem into two smaller parts.

First, to go from A to C, you need to go 4 blocks right and 3 blocks down. This corresponds to the number of arrangements of R R R R D D D.

$$\therefore \text{ Number of paths from } A \text{ to } C = \frac{7!}{4!3!}$$
$$= 35$$

Next, to go from C to B, you need to go 2 blocks right and 4 blocks down. This corresponds to the number of arrangements of R R D D D D.

$$\therefore \text{ Number of paths from } C \text{ to } B = \frac{6!}{2!4!}$$
$$= 15$$

Using the Fundamental Counting Principle,

$$\text{Number of paths from } A \text{ to } B = 35 \times 15$$
$$= 525$$

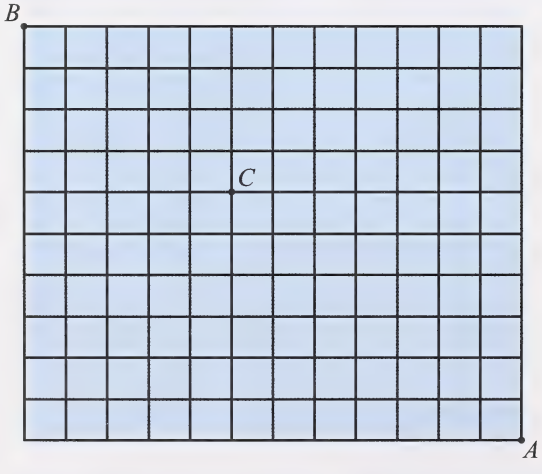
There are 525 paths from A to B.

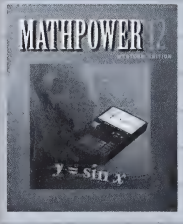
4. In a particular part of town, there are 8 streets that run north-south and 6 avenues that run east-west. Moving only south or west, how many different paths are there from the northeast corner to the southwest corner?

5. Answer the following on pages 352 and 353 of the textbook.

a. questions 2 to 4 of "Practice"
b. questions 5 to 8 of "Applications and Problem Solving"

6. Use the following diagram to answer questions 6.a. and 6.b.





a. Without backtracking, how many paths are there from A to B that pass through C?

b. What percentage of the total number of paths from A to B pass through C?

Turn to page 131 to compare your responses with those in the Appendix.

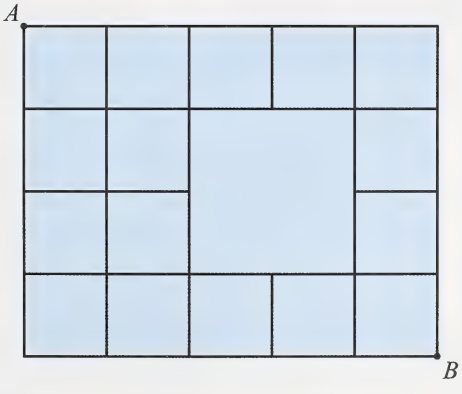
Mathematical Process

- Communication
- Connection
- Estimation
- Mental Math
- Problem Solving
- Reasoning
- Technology
- ☒ Visualization

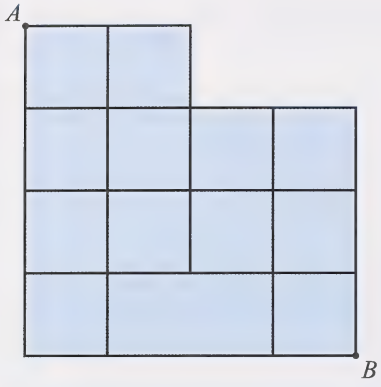
Example

For each diagram, how many paths are there from A to B? Assume there is no backtracking.

a.



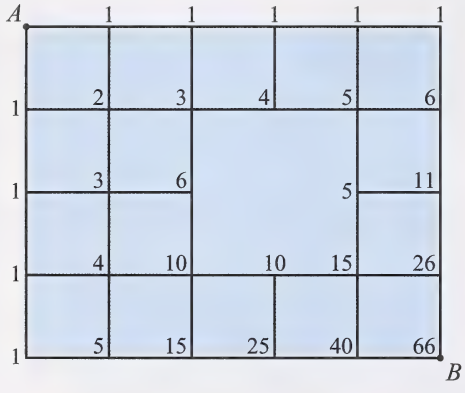
b.



Solution

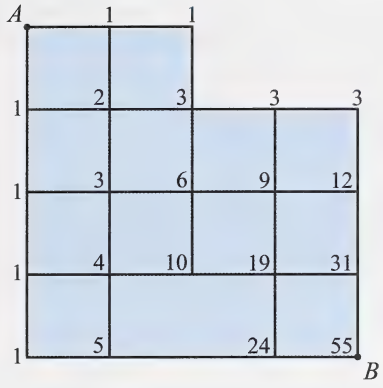
Because of the "holes" within the diagrams, it would be simpler to apply the techniques of Pascal's triangle.

a.



There are 66 paths from A to B.

b.



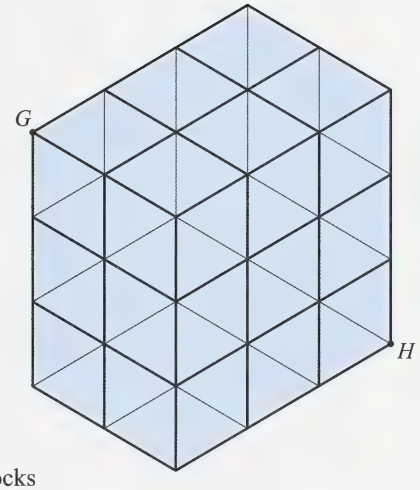
There are 55 paths from A to B.

Mathematical Process

- Communication
- Connection
- Estimation
- Mental Math
- Problem Solving
- [x] Reasoning
- Technology
- Visualization

Example

Determine the number of paths from G to H in the 3-D diagram given on the right. Movement may only occur along the edges toward H.



Solution

In this case if you were to apply the techniques of Pascal's triangle, it would take you a lot of time and effort. So, to solve this problem, it would be better to use distinguishable arrangements.

To go from G to H, you must travel 2 blocks toward the front, 3 blocks right, and 3 blocks down. This is equivalent to the number of distinguishable arrangements of F F R R R D D D.

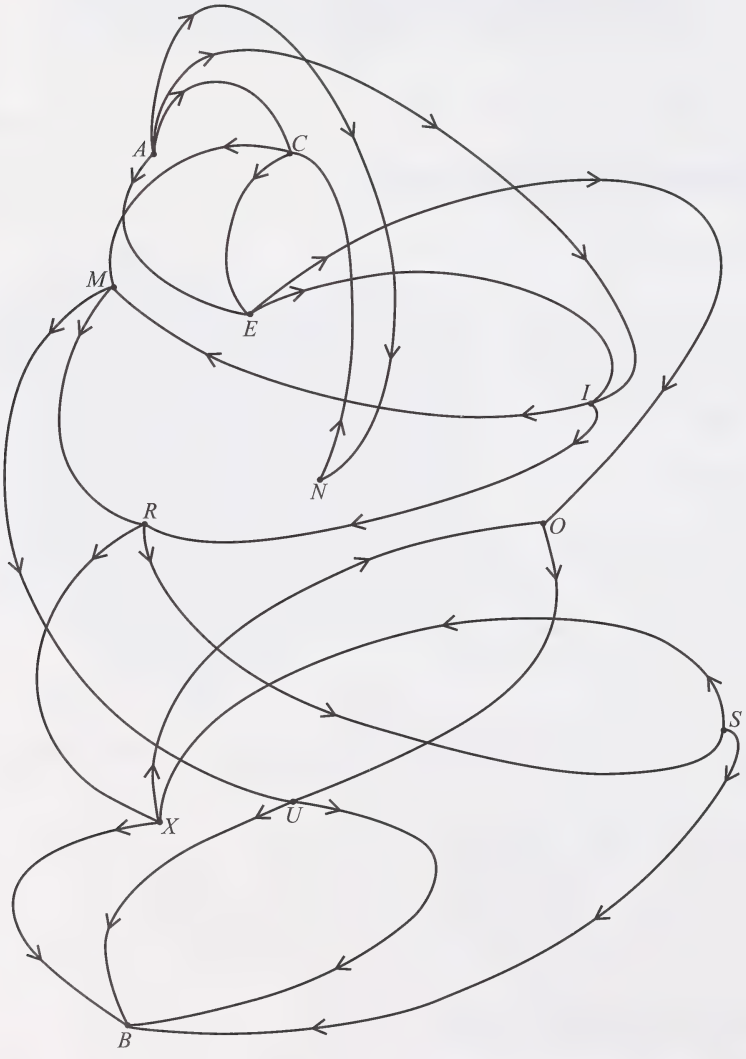
$$\therefore \text{ Number of paths} = \frac{8!}{2!3!3!}$$
$$= \frac{40\,320}{72}$$
$$= 560$$

There are 560 paths from G to H.

Pathway problems aren't restricted to straight line paths. The paths can be curved as well. Although it may look confusing, you still solve this type of problem the same way.

Example

Isabelle and Henrie need to fly from A to B to be at their grandmother's 95th birthday celebration. Their research led them to a number of airlines and connecting flights (as shown in the following diagram). How many different ways can they put together their trip from point A to point B using routes shown?

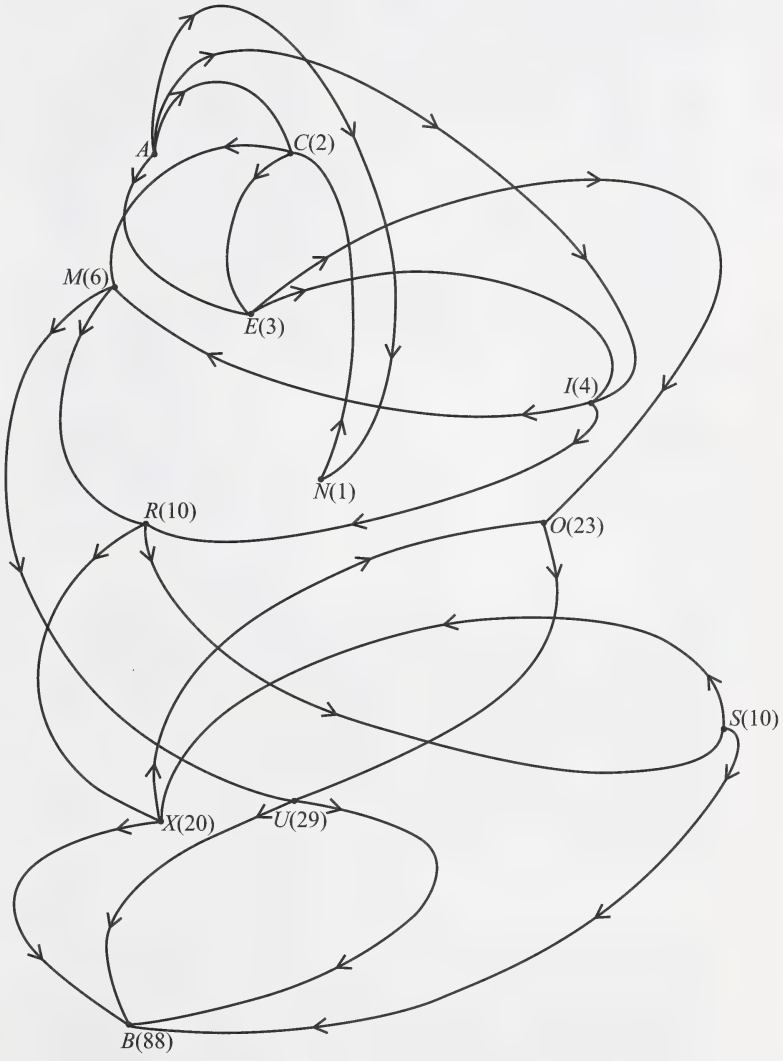


Mathematical Process

- [ ] Communication
- [ ] Connection
- [ ] Estimation
- [ ] Mental Math
- [x] Problem Solving
- [ ] Reasoning
- [ ] Technology
- [x] Visualization

Solution

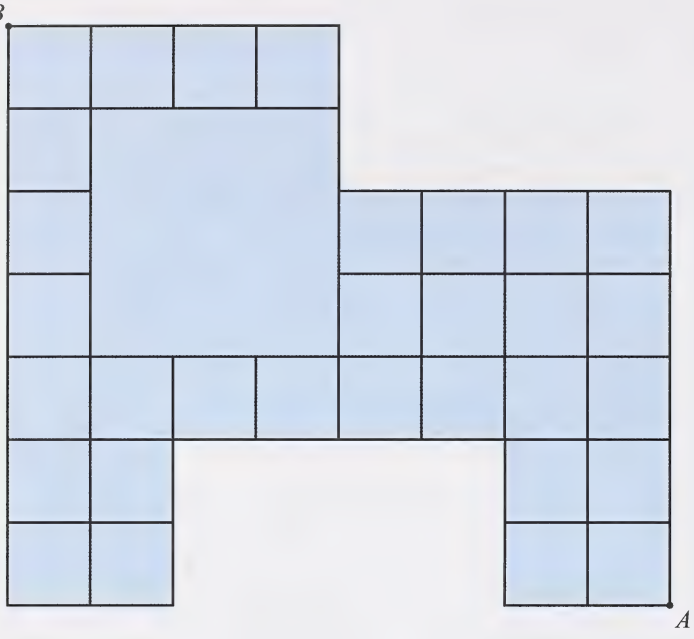
To solve this problem, count the number of paths like you would using Pascal's triangle. For example, there are two ways to leave from A to get to C: a direct flight and one that touches down at N. The complete solution is given.



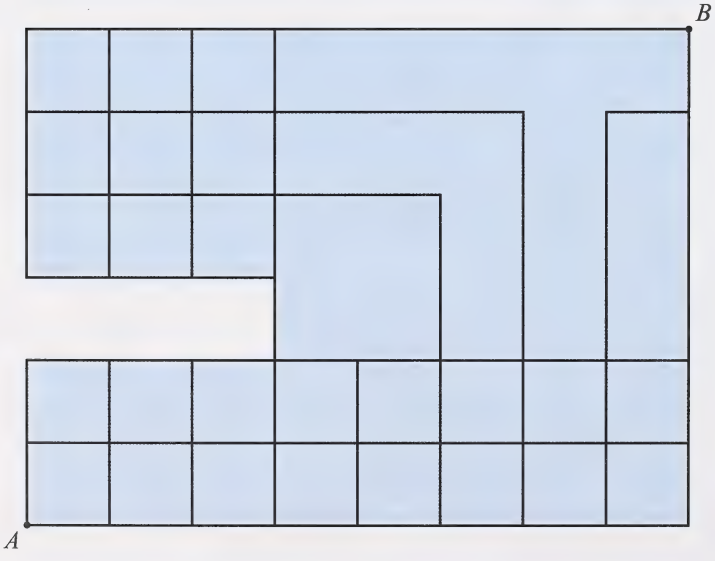
Isabelle and Henrie can choose from 88 different routes to get from point A to point B.

7. For each diagram, determine the number of different paths from A to B. Assume that movement can only occur along the edge towards B.

a.

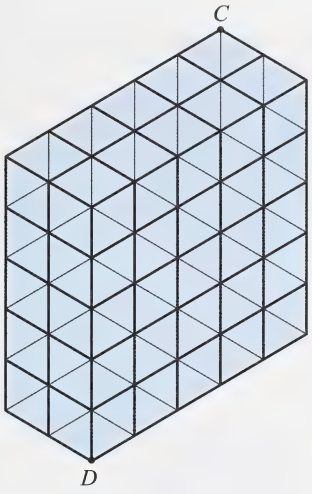


b.

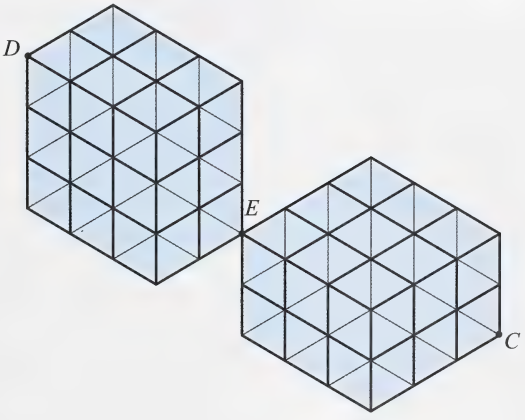


8. For each diagram, determine the number of different paths from C to D. Assume that only movement along the edges toward D.

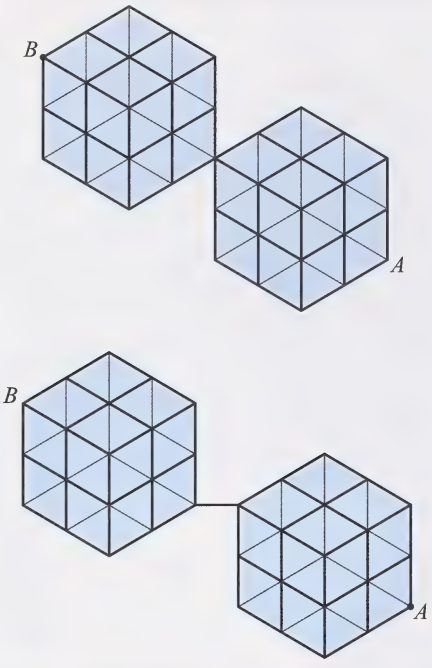
a.



b.



9. In the two diagrams given, will the number of paths from A to B differ?



Turn to page 141 to compare your responses with those in the Appendix.

Now Try This

You have heard of numbers that are squares and cubes, but have you heard of **triangular numbers**? Did you know that triangular numbers occur in Pascal's triangle?

The following diagram shows the first three triangular numbers.

The nth triangle is formed by adding n dots to the bottom of the $(n-1)$ triangle. The triangular number is found by counting the number of dots that are required to form the triangle.

Mathematical Process

- ☐ Communication
- ☐ Connection
- ☐ Estimation
- ☐ Mental Math
- ☒ Problem Solving
- ☐ Reasoning
- ☐ Technology
- ☒ Visualization

10. Explore triangular numbers by completing the following questions.

a. Draw the fourth and fifth triangular numbers.

b. What are the values of the fourth and fifth triangular numbers?

c. Complete the first eight rows of Pascal's triangle. Recall that Pascal's triangle starts with the first row as Row 0, the second row as Row 1, and so on. Circle the triangular numbers that appear.

d. Using Pascal's triangle, predict the value of the sixth triangular number. Check your prediction by drawing the number and counting the dots.

e. Write the first eight triangular numbers as combinations using the symbol ${}_nC_r$. Write the general expression for a triangular number.

Turn to page 144 to compare your responses with those in the Appendix.

Mathematical Process

- [x] Communication
- [ ] Connection
- [ ] Estimation
- [ ] Mental Math
- [ ] Problem Solving
- [ ] Reasoning
- [x] Technology
- [ ] Visualization

Looking Back

In this activity you explored pathway problems, Pascal's triangle, and the relationship between them.

You barely scratched the surface of the multitude of applications of Pascal's triangle in this activity. Use your research skills and the Internet to explore Pascal's triangle further. Try starting at the following website:

http://www.mathforum.org/dr.math/faq/faq.pascal.triangle.html

Record your finding in your journal.

Activity 2: The Binomial Theorem

When planning a vacation in which you'll be driving to your destination, you may choose between two different routes: a scenic route or a shortcut.

When expanding powers of binomials, such as $(x + y)^8$, you may wish to include every detail, showing how the terms are multiplied together one after the other—the scenic route. However, you may wish to use the **Binomial Theorem**—the shortcut. The Binomial Theorem is a quick method of obtaining the terms in the expansion of the power of a binomial.

In this activity you will derive and apply the Binomial Theorem. You will also explore the relationship between the Binomial Theorem and Pascal's triangle.

Turn to page 354 of *MATHPOWER 12* and read "The Binomial Theorem."

1. Answer the following on page 354 of the textbook.

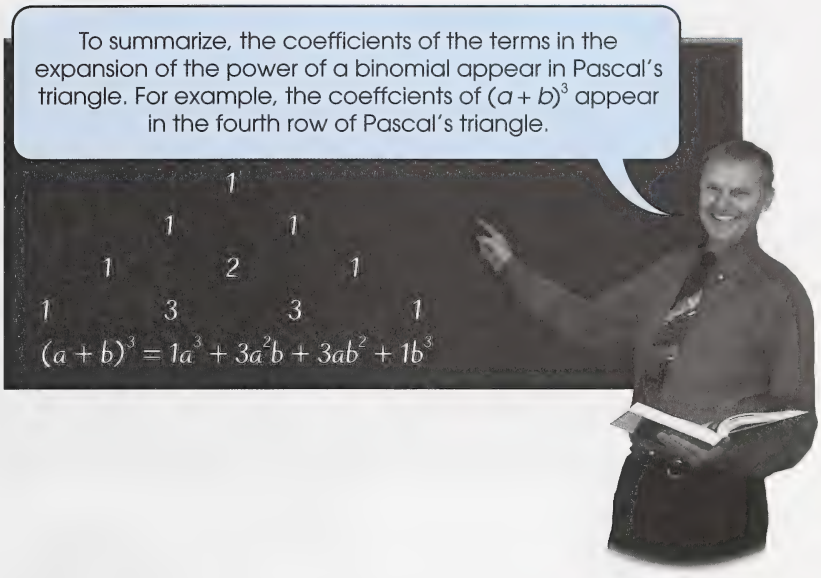
   a. "Explore: Look for a Pattern"
   b. questions 1 to 5 of "Inquire"

Turn to page 145 to compare your responses with those in the Appendix.

Binomial Theorem: a theorem used to determine the expansion of the power of a binomial

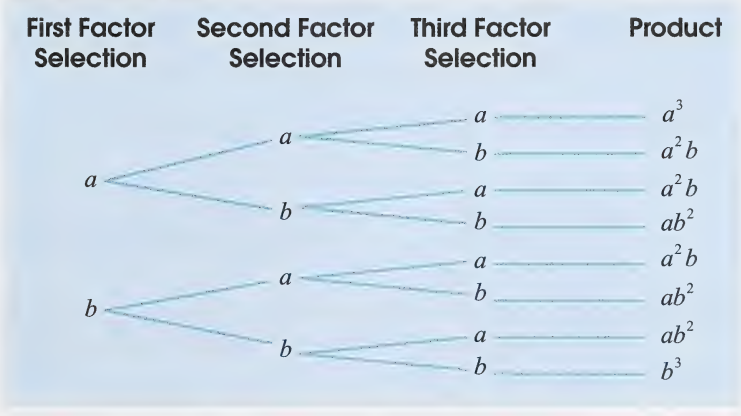
Mathematical Process

- Communication
- [X] Connection
- Estimation
- Mental Math
- Problem Solving
- Reasoning
- Technology
- Visualization


To summarize, the coefficients of the terms in the expansion of the power of a binomial appear in Pascal's triangle. For example, the coeffcients of $(a+b)^3$ appear in the fourth row of Pascal's triangle.
1
1 1
1 2 1
1 3 3 1
$(a+b)^3 = 1a^3 + 3a^2b + 3ab^2 + 1b^3$

Recall from Activity 1 that the entries in Pascal's triangle can be written as combinations. For example, the coefficients 1, 3, 3, and 1 can be written as ${}_3C_0$, ${}_3C_1$, ${}_3C_2$, and ${}_3C_3$ respectively.

Why are the coefficients combinations? To answer this question, examine how $(a+b)^3$ or $(a+b)(a+b)(a+b)$ is obtained. When the three factors of $(a+b)$ are multiplied, you form their product by first selecting an a or a b from each factor and then multiplying your selections together. Study the following tree diagram.



The coefficient of a^3 is obtained by selecting 0 bs from the three factors. Therefore, the coefficient of a^3 is ${}_3C_0 = 1$.

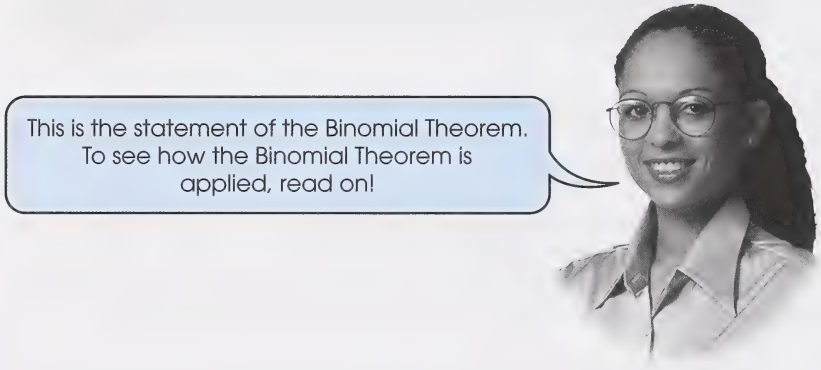
The coefficient of a^2b is obtained by selecting 1 b from the three factors. This can be done 3 ways. Therefore, the coefficient of a^2b is ${}_3C_1 = 3$.

The coefficient of ab^2 is obtained by selecting 2 bs from the three factors. This can be done 3 ways. Therefore, the coefficient of a^2b is ${}_3C_2 = 3$.

The coefficient of b^3 is obtained by selecting 3 bs from the three factors. Therefore, the coefficient of b^3 is ${}_3C_3 = 1$.

In general, the coefficient of each term is the number of ways of selecting the appropriate number of bs from the factors.

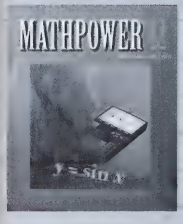
$$\therefore (a+b)^n = {}_nC_0a^n + {}_nC_1a^{n-1}b + {}_nC_2a^{n-2}b^2 + {}_nC_3a^{n-3}b^3 + \ldots + {}_nC_nb^n$$

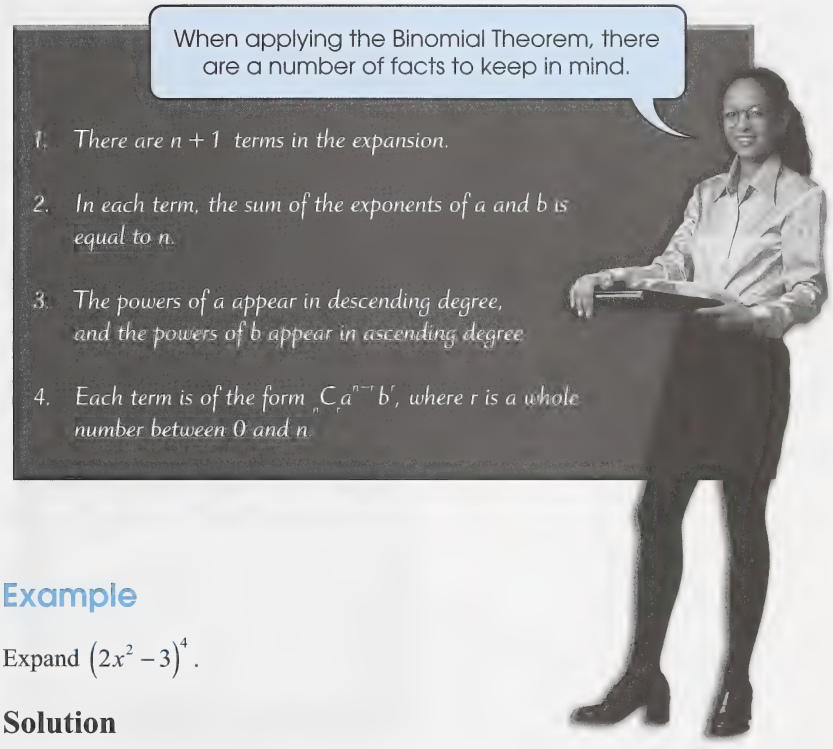


Turn to page 355 of *MATHPOWER 12* and work through Example 1.

2. Answer questions 1, 2, 5, and 10 of "Practice" on page 356 of the textbook.

Turn to page 147 to compare your responses with those in the Appendix.




When applying the Binomial Theorem, there are a number of facts to keep in mind.
1. There are $n+1$ terms in the expansion.
2. In each term, the sum of the exponents of a and b is equal to n.
3. The powers of a appear in descending degree, and the powers of b appear in ascending degree
4. Each term is of the form ${}_nC_r a^{n-r} b^r$, where r is a whole number between 0 and n

Example

Expand $\left(2x^2-3\right)^4$.

Solution

Use $(a+b)^4 = {}_4C_0a^4 + {}_4C_1a^3b + {}_4C_2a^2b^2 + {}_4C_3ab^3 + {}_4C_4b^4$.

Substitute $2x^2$ for a and -3 for b, and simplify.

$$\begin{aligned}\left(2x^2-3\right)^4 &= {}_4C_0\left(2x^2\right)^4 + {}_4C_1\left(2x^2\right)^3(-3) + {}_4C_2\left(2x^2\right)^2(-3)^2 \\ &\quad + {}_4C_3\left(2x^2\right)(-3)^3 + {}_4C_4(-3)^4 \\ &= 1\left(16x^8\right) + 4\left(8x^6\right)(-3) + 6\left(4x^4\right)(9) + 4\left(2x^2\right)(-27) + 1(81) \\ &= 16x^8 - 96x^6 + 216x^4 - 216x^2 + 81\end{aligned}$$

In the preceding example, notice that the signs of the terms alternate. This occurs whenever the signs of the two terms in the original binomial differ.

In the next three examples you will use the formula for the general term, $t_{r+1} = {}_nC_r a^{n-r} b^r$. By letting $r = 0, 1, 2, \ldots$, you can construct any term in the expansion. You must note that ${}_nC_r a^{n-r} b^r$ determines the $(r + 1)$th term. For example, in the Binomial Theorem, ${}_nC_3$ is the coefficient of the fourth $(3 + 1)$ term.

Example

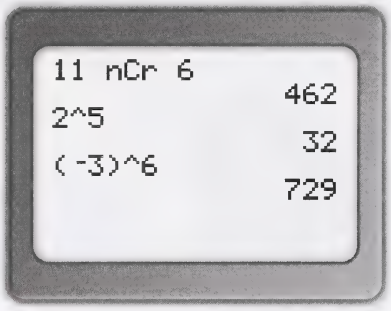
Determine the seventh term, t_7, in the expansion of $(2x - 3)^{11}$.

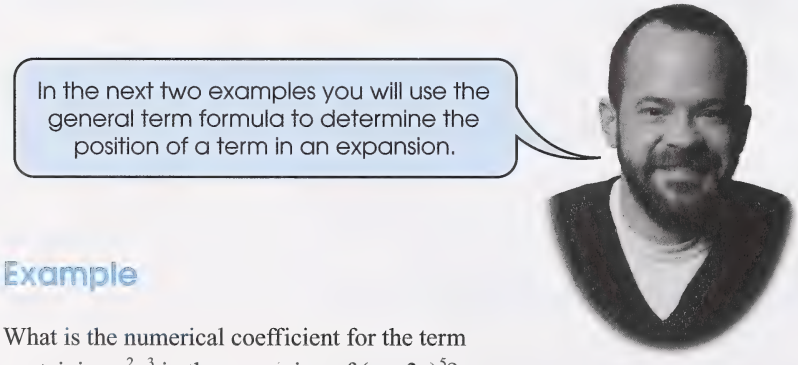
Solution

The seventh term is generated by substituting $r = 6$ into the general term formula.

$n = 11$, $a = 2x$, $b = -3$, and $r = 6$

$$\therefore t_{r+1} = {}_nC_r a^{n-r} b^r$$
$$t_{6+1} = {}_{11}C_6 (2x)^{11-6} (-3)^6$$
$$t_7 = 462(2x)^5 (-3)^6$$
$$= 462(32x^5)(729)$$
$$= 10\,777\,536x^5$$





Example

What is the numerical coefficient for the term containing x^2y^3 in the expansion of $(x - 3y)^5$?

Mathematical Process

- Communication
- Connection
- Estimation
- Mental Math
- [x] Problem Solving
- [x] Reasoning
- Technology
- Visualization

Solution

The exponent of y in x^2y^3 is 3.

$\therefore r = 3, n = 5, a = x$, and $b = -3y$

$$\therefore t_{r+1} = {}_nC_r a^{n-r} b^r$$
$$t_{3+1} = {}_5C_3 (x)^{5-3} (-3y)^3$$
$$t_4 = 10(x^2)(-27y^3)$$
$$= -270x^2y^3$$

The numerical coefficient for the term containing x^2y^2 is -270.

Mathematical Process

- [ ] Communication
- [ ] Connection
- [ ] Estimation
- [ ] Mental Math
- [x] Problem Solving
- [x] Reasoning
- [ ] Technology
- [ ] Visualization

Example

Which term in the expansion of $(x^2 + y)^{15}$ contains x^6?

Solution

Using the formula $t_{r+1} = {}_nC_r a^{n-r} b^r$, the required term is of the form $t_{r+1} = {}_{15}C_r x^6 y^r$.

Because the first term, a, is x^2,

$$t_{r+1} = {}_{15}C_r (x^2)^3 y^r$$

Because the sum of the exponents of the powers of the two terms in the binomial is 15, find r.

$$r + 3 = 15$$
$$r = 12$$

$$\therefore t_{r+1} = t_{12+1}$$
$$= t_{13}$$

The thirteenth term contains x^6.

Turn to page 356 of *MATHPOWER 12* and work through Example 2.

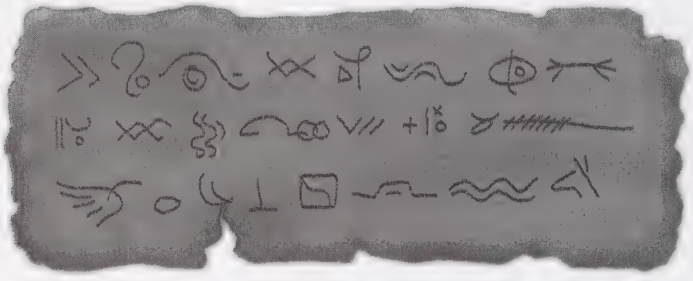
3. Answer the following on pages 356 and 357 of the textbook.

   a. questions 12, 13, 15, 16, 27, 28, 29, and 30 of "Practice"
   b. questions 31, 33, 36, 37, and 39 of "Applications and Problem Solving"

Turn to page 148 to compare your responses with those in the Appendix.

Now Try This

In this activity you used the patterns in the Binomial Theorem and in Pascal's triangle. Some patterns, though, are not as obvious as these. Coded messages contain patterns, but these patterns are deliberately designed to be difficult to decipher.



Turn to "CONNECTING MATH AND COMPUTERS" on page 358 of *MATHPOWER 12* and read "Cryptography."

4. Answer questions 2.a. and 2.b. of Investigation 2, "Classification by Mathematical Operations" on page 359 of the textbook.

Turn to page 157 to compare your responses with those in the Appendix.

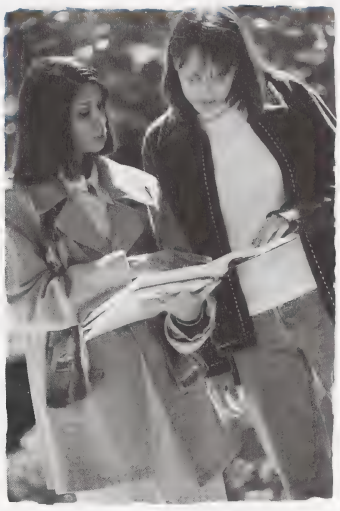
Mathematical Process

- Communication
- [x] Connection
- Estimation
- Mental Math
- Problem Solving
- Reasoning
- Technology
- Visualization

Mathematical Process

- [x] Communication
- [ ] Connection
- [ ] Estimation
- [ ] Mental Math
- [ ] Problem Solving
- [ ] Reasoning
- [ ] Technology
- [ ] Visualization

Looking Back



In this activity you used the Binomial Theorem and Pascal's triangle to expand powers of binomials. You then used the formula for the general term to find specific terms in an expansion.

In your journal, expand $(2x + 3)^3$ using the Binomial Theorem. Time yourself to see how long it takes. Then expand $(2x + 3)^3$ by multiplying term by term. Time yourself again. Which method was faster? Which method do you prefer? Exchange information with another student taking Pure Mathematics 30. Which method did he or she prefer?

Follow-up Activities

If you had difficulties understanding the concepts and skills in the activities, it is recommended that you do the Extra Help. If you have a clear understanding of the concepts and skills, it is recommended that you do the Enrichment. You may decide to do both.

Extra Help

A common mistake many students make when applying the Binomial Theorem is breaking one of the exponent rules. An example of this mistake is writing $\left(2x^2\right)^4$ as $8x^8$.

There are some simple strategies you can do to avoid these errors.

Remember: Multiplication represents repeated addition, and **powers** represent repeated multiplication.

$2 \times 4 = 2 + 2 + 2 + 2$ $\quad$ $2^4 = 2 \times 2 \times 2 \times 2$

If you remind yourself that powers are short forms for repeated multiplication, you are less likely to make errors!

Mathematical Process

- Communication
- Connection
- Estimation
- [X] Mental Math
- Problem Solving
- Reasoning
- Technology
- Visualization

Example

Simplify $(2x)^5$.

Solution

Read this question to yourself as you would if you were reading it out loud:

"Simplify the expression $2x$ all to the fifth."

The word "all" emphasizes that the base of the power is $2x$ and that

$$\begin{aligned}(2x)^5 &= (2x)(2x)(2x)(2x)(2x) \\ &= 32x^5\end{aligned}$$

Another common mistake is to ignore the importance of brackets. Work through the following example.

Example

Simplify the following.

a. $(-5)^0$ **b.** -5^0

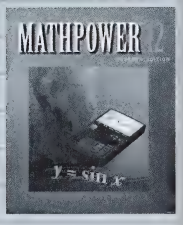
Solution

a. The brackets in $(-5)^0$ indicate that the base of the power is -5. Therefore, think of this expression as "the base -5 raised to the exponent 0." Remember, any non-zero base raised to the exponent 0 is equal to 1.

$\therefore (-5)^0 = 1$

b. There are no brackets in the expression -5^0. The base of the power is not -5. If brackets are inserted, they do not include the negative sign!

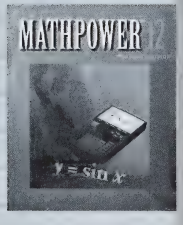
$$\therefore \ -5^0 = -\left(5^0\right)$$
$$= -1$$



1. Turn to page 333 of *MATHPOWER 12* and answer questions 1, 2, 3, 4, 6, 8, 13, and 14 of "Mental Math."

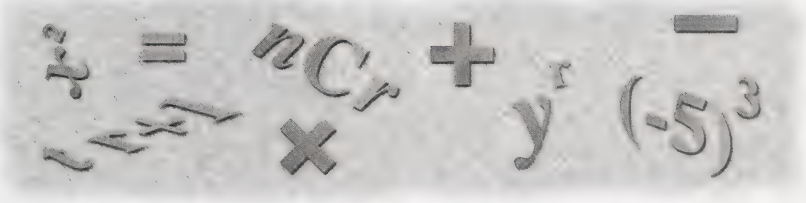
Turn to page 158 to compare your responses with those in the Appendix.

The following questions will allow you to practise applying the exponent rules while applying the Binomial Theorem. There are also more pathway questions.



2. Turn to pages 360 and 361 of *MATHPOWER 12* and answer questions 9, 10, 11, 12.a., 12.d., 13.b., and 13.d. of "Review."

Turn to page 159 to compare your responses with those in the Appendix.



Enrichment

You have applied Pascal's triangle numerous times throughout this section when expanding powers of binomials and solving pathway problems. Did you know that you can use Pascal's triangle to evaluate powers of 11?

Look at the following pattern.

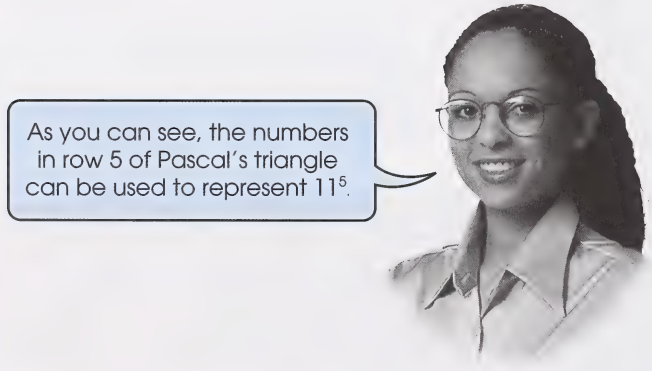
Powers of 11		Pascal's Triangle
$11^0 = 1$	row 0	1
$11^1 = 11$	row 1	1 1
$11^2 = 121$	row 2	1 2 1
$11^3 = 1331$	row 3	1 3 3 1
$11^4 = 14\,641$	row 4	1 4 6 4 1

Note: The rows are named to correspond to the exponent of the power of 11.

Does the pattern continue?

The value of $11^5 = 161\,051$. These digits are not the numbers in row 5, which are 1, 5, 10, 10, 5, and 1. However,

$$\begin{aligned} &1(10)^5 + 5(10)^4 + 10(10)^3 + 10(10)^2 + 5(10)^1 + 1(10)^0 \\ &= 100\,000 + 50\,000 + 10\,000 + 1000 + 50 + 1 \\ &= 161\,051 \\ &= 11^5 \end{aligned}$$



Example

Can the number in row 6 of Pascal's triangle be used to represent 11^6?

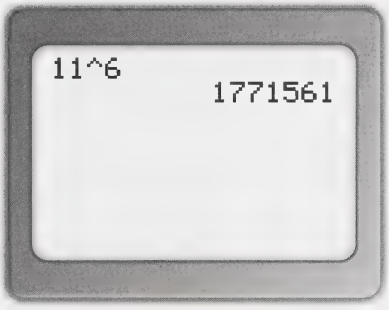
Solution

Row 6 of Pascal's triangle contains the numbers 1, 6, 15, 20, 15, 6, and 1.

Using these numbers and the descending powers of 10, starting with 10^6, you get the following:

$$\begin{aligned}
&1(10)^6 + 6(10)^5 + 15(10)^4 + 20(10)^3 + 15(10)^2 + 6(10)^1 + 1(10)^0 \\
&= 1\,000\,000 + 600\,000 + 150\,000 + 20\,000 + 1500 + 60 + 1 \\
&= 1\,771\,561 \\
&= 11^6
\end{aligned}$$

Check



Yes, row 6 of Pascal's triangle can be used to represent 11^6.

1. a. Which power of 11 can be determined by using row 8 of Pascal's triangle? Evaluate this power, and check your answer using a graphing calculator.

b. Write 11^8 as a power of a binomial. Expand this binomial using the Binomial Theorem.

Turn to page 163 to compare your responses with those in the Appendix.

Another application of Pascal's triangle involves using certain coefficients to form a special class of polynomials, called Chebyshev Polynomials.

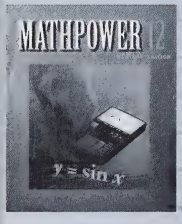
Turn to page 361 of *MATHPOWER 12* and read "Exploring Math: Chebyshev Polynomials."

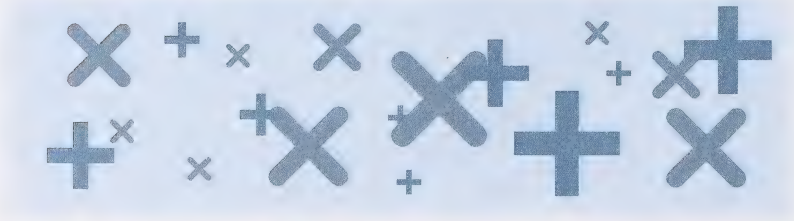
2. Answer questions 1 to 3 of "Exploring Math: Chebyshev Polynomials" on page 361 of the textbook.

Turn to page 164 to compare your responses with those in the Appendix.

Mathematical Process

- Communication
- Connection
- Estimation
- Mental Math
- Problem Solving
- Reasoning
- [x] Technology
- Visualization





Conclusion

You began this section by analyzing pathway problems. As part of your analysis, you developed and explored Pascal's triangle. You then discovered that you can use Pascal's triangle to solve pathway problems and, together with the Binomial Theorem, expand powers of the form $(a + b)^n$, where n is a whole number.

This section emphasized the close relationship among pathway problems, Pascal's triangle, and the Binomial Theorem. Your study of these three topics is like trying to work your way out of a maze. Regardless of which of these topics you choose, you will either encounter the other two in the course of your work or get totally lost!

CORBIS

Hedge Maze in Giverny, France

For many people, a maze is the ultimate pathway problem. Hedge mazes, like the one in the photograph, are often incorporated as part of the overall design of large formal gardens. These mazes provide a place for privacy and contemplation as well as for novelty and entertainment.

Assignment

Turn to Assignment Booklet 6B and complete the assignment for Section 2.

Module Summary

This module introduced you to a branch of mathematics called combinatorics—the study of permutations, combinations, and their application to a variety of seemingly diverse problem situations, such as pathway problems and the powers of binomials.

In Section 1 you explored the Fundamental Counting Principle. Using this principle, you developed techniques for solving problems involving permutations (or arrangements) of entire sets of objects, permutations of subsets of these objects, and distinguishable permutations. You then investigated combinations or selections for which the order in which the objects are selected does not matter. Throughout your analysis of combinatorics, you were presented with problems involving several cases or restrictions.

In Section 2 you used your knowledge of combinatorics to address pathway problems, Pascal's triangle, the Binomial Theorem, and the relationships among them.

The study of combinatorics arises out of the need to determine numbers of possibilities of performing tasks, such as the number of licence plates that can be manufactured, the number of possibilities for carrying out a renovation or construction project successfully, and, simply, the number of different routes cyclists can take on the trails near Banff, Alberta.

Final Module Assignment

Turn to Assignment Booklet 6B and complete the final module assignment. Remember to submit your completed Assignment Booklet for assessment.

Glossary

Calculator Functions

Suggested Answers

Credits

Glossary

Binomial Theorem: a theorem used to determine the expansion of the power of a binomial

combination: a selection of all or part of a set of objects, where the order of the selection is not important

combinatorics: the branch of mathematics that includes the study of permutations and combinations

factorial notation: the notation, $n!$, used to represent the product of the first n natural numbers

Fundamental Counting Principle: the principle for determining the number of ways two or more operations can be performed together

If one operation can be performed in m ways and a second in n ways, together they can be performed in mn ways.

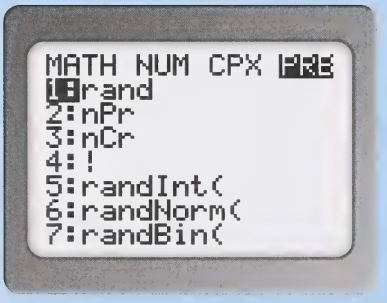
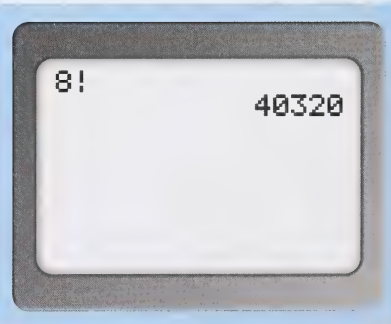
Pascal's triangle: the triangular arrangement of numbers, where 1 is in the first row and each number in the succeeding rows is the sum of the two numbers above it in the preceding row

pathway problem: a problem involving the number of ways of going from one point to another

permutation: an arrangement of all or part of a set of objects, where the order of the arrangement is important

tree diagram: a diagram used to illustrate the number of ways several related tasks can be performed

Calculator Functions

Calculator Function	Displays
Calculating Factorials e.g., 8! 8 MATH [Select the PRB menu.] 4 (4!) ENTER	 

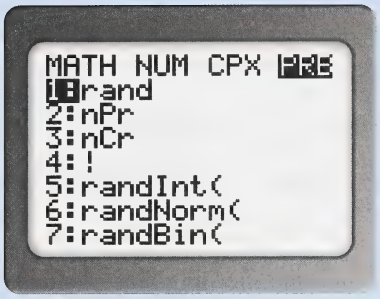
Calculating Permutations

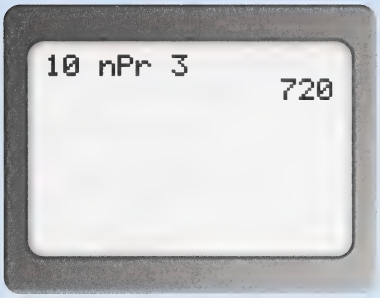
e.g., ${}_{10}P_3$

1 0 MATH

[Select the PRB menu.]

2 (2:nPr) 3 ENTER





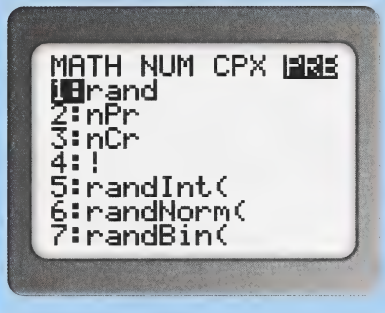
Calculating Combinations

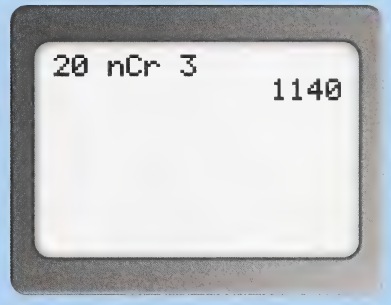
e.g., ${}_{20}C_3$

2 0 MATH

[Select the PRB menu.]

3 (3:nCr) 3 ENTER

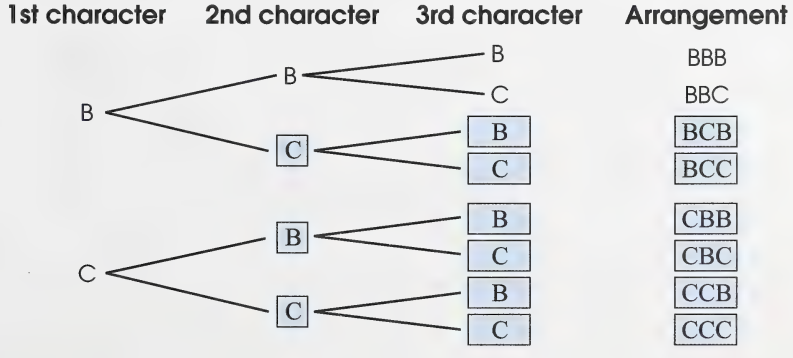




Suggested Answers

Section 1: Activity 1

1. a. Textbook question "Explore: Draw a Tree Diagram," p. 334

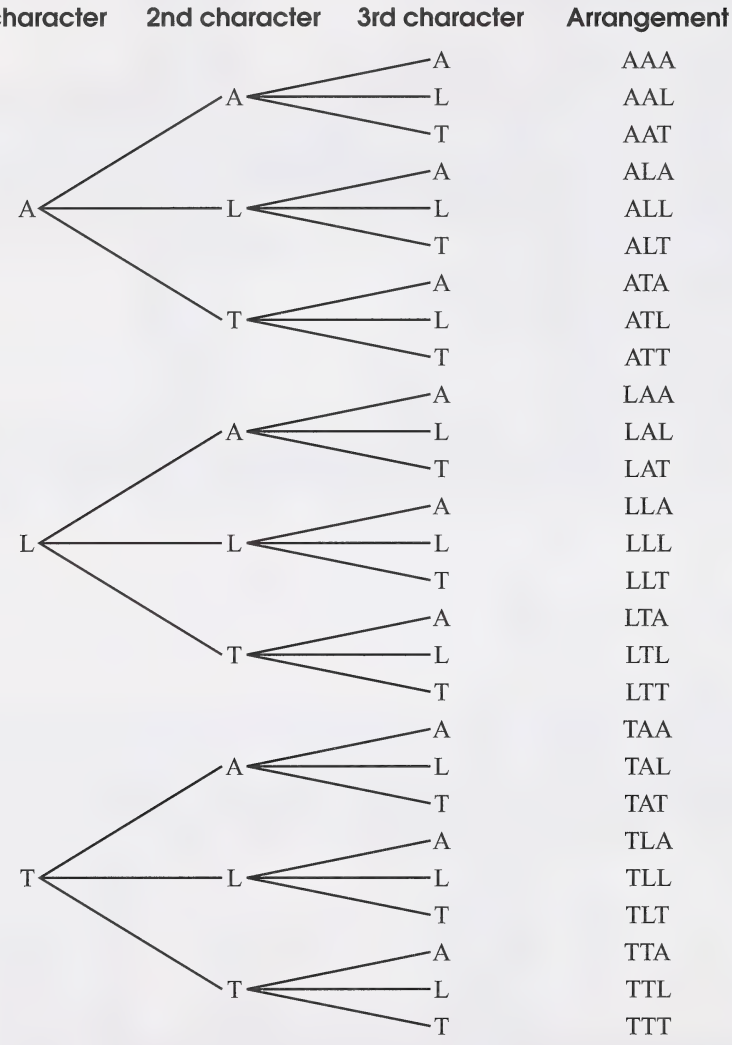


Section 1: Activity 1 (continued)

b. Textbook questions 1 to 4 of "Inquire," pp. 334 and 335

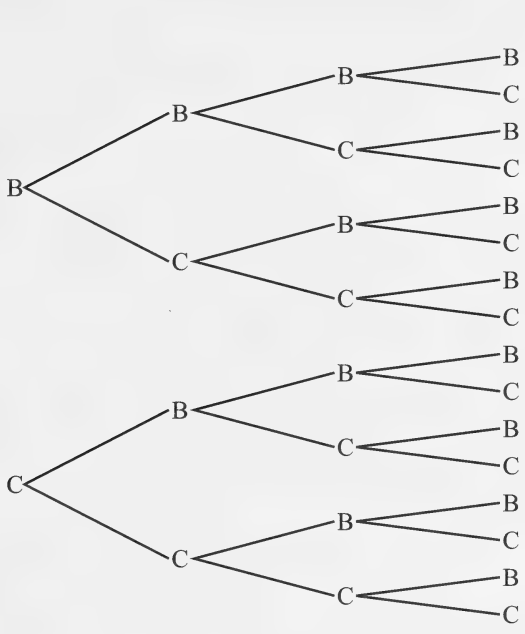
1. There are 8 ways to arrange the three letters if only Bs and Cs are used.

2.



There are 27 three-letter arrangements.

3. a. A tree diagram can be used to show the arrangements of 4 letters if only the letters B and C are used.



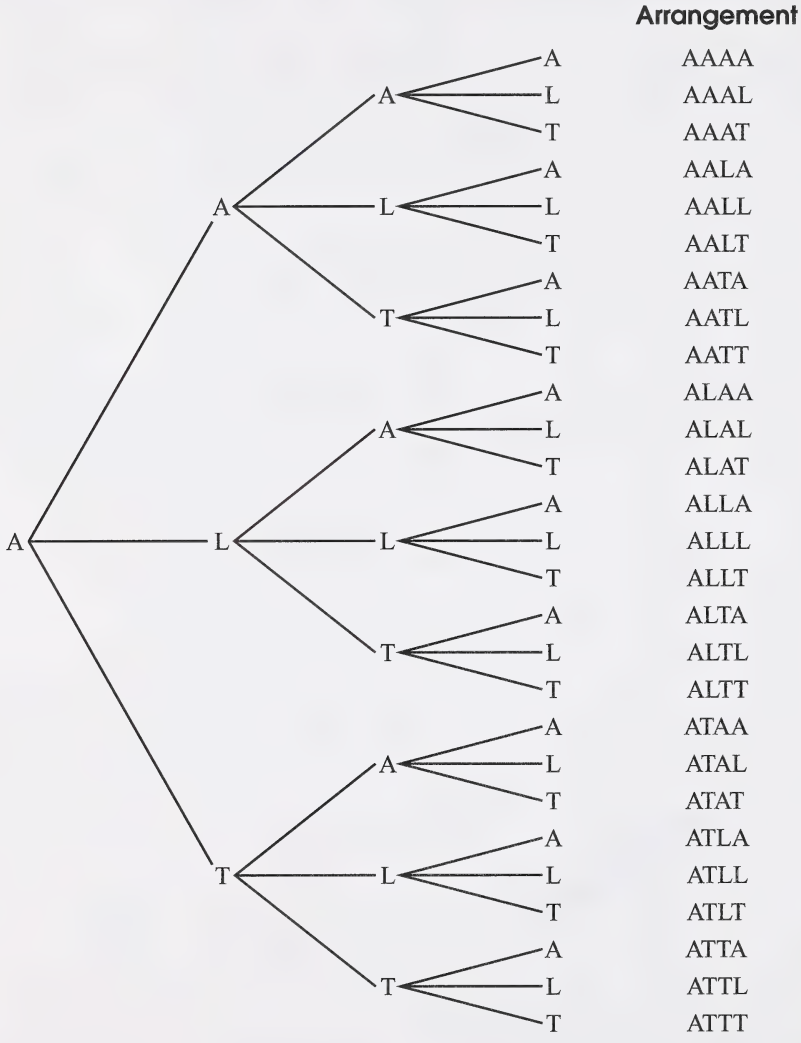
Arrangement
BBBB
BBBC
BBCB
BBCC
BCBB
BCBC
BCCB
BCCC
CBBB
CBBC
CBCB
CBCC
CCBB
CCBC
CCCB
CCCC

There are 16 ways to arrange the four letters if only B and C are used.

Section 1: Activity 1 (continued)

b. A partial tree diagram can be used to determine the number of ways to arrange four letters if only A, L, and T are used.

The branches for L and T will have the same number of arrangements as the branch for A.

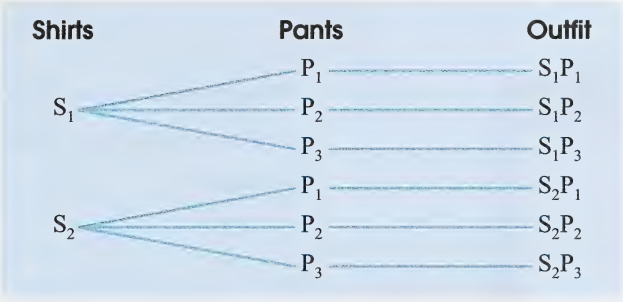


There are 27 ways to arrange A, L, and T beginning with A. Because there are three branches altogether, there will be $27 \times 3 = 81$ ways to arrange the four letters if only the letters A, L, and T are used.

4. The number of arrangements can be determined using the formula k^n, where k is the number of letters that can be used in the arrangement and n is the number of letters in the arrangement. In question 3.a. it is 2^4 because only two letters (B and C) are used in a four-letter arrangement. In question 3.b. it is 3^4 because only three letters (A, L, and T) are used in a four-letter arrangement.

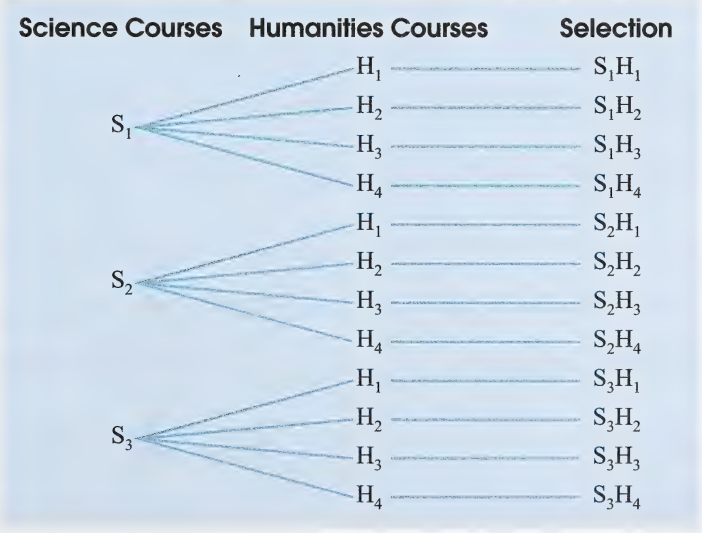
2. Textbook questions 1 and 3 of "Practice," p. 336

1. Let the two shirts be S_1 and S_2, and let the three pairs of pants be P_1, P_2, and P_3.



There are 6 different outfits that can be made.

3. Let S_1 and S_2 be the science courses, and let H_1, H_2, H_3, and H_4 be the humanities courses.



There are 12 possible course selections.

Section 1: Activity 1 (continued)

3. Textbook questions 5, 6, and 9 to 14 of "Applications and Problem Solving," pp. 336 and 337

5. The first stage, choosing the first letter, has 26 choices. The second, third, and fourth stages also have 26 possible choices. The fifth stage, choosing the first number, has 10 choices. The sixth and seventh stages also have 10 possible choices.

$$\begin{aligned}\therefore \text{ Number of possible licence plates} &= 26\times26\times26\times26\times10\times10\times10\\ &= 26^4\times10^3\\ &= 456\,976\,000\end{aligned}$$

In Ontario, 456 976 000 – 17 576 000 = 439 400 000 more licence plates can be issued using a seven-character format.

6. a. There are 4 choices (A, C, G, or T) for all three bases.

$$\begin{aligned}\therefore \text{ Number of genetic codes} &= 4\times4\times4\\ &= 4^3\\ &= 64\end{aligned}$$

b. There are 4 choices (A, C, G, or T) for all four bases.

$$\begin{aligned}\therefore \text{ Number of genetic codes} &= 4\times4\times4\times4\\ &= 4^4\\ &= 256\end{aligned}$$

c. There are 4 choices for each of the n bases.

$$\begin{aligned}\therefore \text{ Number of genetic codes} &= \underbrace{4\times4\times\ldots\times4}_{n \text{ times}}\\ &= 4^n\end{aligned}$$

9. a. For a single burst,

$$\text{Number of letters coded} = 2$$

For two bursts,

$$\begin{aligned}\text{Number of letters coded} &= 2\times2\\ &= 4\end{aligned}$$

For three bursts,

$$\begin{aligned}\text{Number of letters coded} &= 2\times2\times2\\ &= 8\end{aligned}$$

For four bursts,

$$\begin{aligned}\text{Number of letters coded} &= 2\times2\times2\times2\\ &= 16\end{aligned}$$

b. In total, 2 + 4 + 8 + 16 = 30 characters can be coded by up to four bursts.

10. a. Each of the 5 parts has 4 possible patterns.

$$\therefore \text{Number of different keys} = 4 \times 4 \times 4 \times 4 \times 4$$
$$= 4^5$$
$$= 1024$$

b. Let p be the number of patterns.

Because there are 6 parts,

$$\text{Number of different keys} = p \times p \times p \times p \times p \times p$$
$$= p^6$$

$$\therefore \text{Number of different keys} \geq 10\,000$$
$$p^6 \geq 10\,000$$
$$p \geq \sqrt[6]{10\,000}$$
$$p \geq 4.6$$

The number of patterns, p, must be a natural number.
Therefore, 5 patterns are needed for each part.

c. If the number of parts was increased by one (to 6), the number of different keys would be

$$5 \times 5 \times 5 \times 5 \times 5 \times 5 = 5^6$$
$$= 15\,625$$

If the number of patterns was increased by one (to 6), the number of different keys would be

$$6 \times 6 \times 6 \times 6 \times 6 = 6^5$$
$$= 7776$$

Therefore, modifying the number of parts would produce a greater number of different keys.

11. a. From Regina to Calgary, there are 4 modes of transportation.

From Calgary to Vancouver, there are 3 modes of transportation.

$$\therefore \text{Number of ways to travel} = 4 \times 3$$
$$= 12$$

There are 12 different ways to travel from Regina to Vancouver.

Section 1: Activity 1 (continued)

b. There are 12 different ways to travel from Regina to Vancouver and 2 different ways to travel from Vancouver to Regina.

$$\therefore \text{ Number of ways to travel} = 12 \times 2$$
$$= 24$$

There are 24 different ways to make the entire journey.

12. a. There are 8 choices for the first digit, 2 choices for the second digit, and 9 choices for the third digit.

$$\therefore \text{ Number of different area codes} = 8 \times 2 \times 9$$
$$= 144$$

b. There are 8 choices for the first digit and 10 choices for both the second and third digits.

$$\therefore \text{ Number of different area codes} = 8 \times 10 \times 10$$
$$= 800$$

13. There are 256 levels of intensity for each red, green, and blue dot in a pixel.

$$\therefore \text{ Number of colours possible} = 256 \times 256 \times 256$$
$$= 256^3$$
$$= 16\,777\,216$$

14. a. The following represents each licence plate.

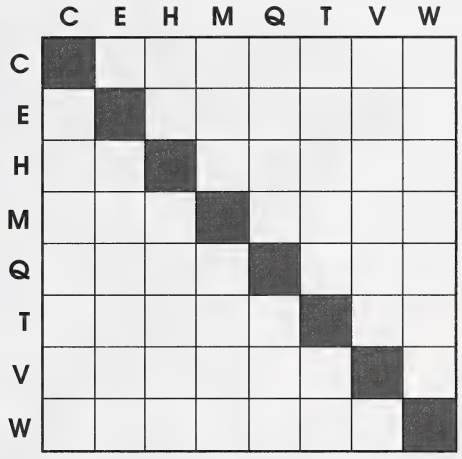
There is only 1 choice, B, for Box 1. There are 26 choices for Box 2 and for Box 3. There are 10 choices each for Box 4, Box 5, and Box 6.

$$\therefore \text{ Number of licence plates} = 1 \times 26 \times 26 \times 10 \times 10 \times 10$$
$$= 1 \times 26^2 \times 10^3$$
$$= 676\,000$$

b. Answers may vary. The population of New Brunswick in 2001 was 729 498. The licence plates may need to be changed in 10 years or so. At that time, New Brunswick may need to remove the restriction on the first letter of its licence plate.

4. Textbook question "LOGIC POWER," p. 337

Using the first letter of each Canadian city to represent that city, draw the following diagram.



Sportscaster A predicted H, W, E, and V to win; sportscaster B predicted Q, W, M, and E to win; and sportscaster C predicted E, M, V, and C to win. Notice that E was chosen by all three sportscasters; therefore, E could not have played H, W, V, Q, M, or C. Therefore, Edmonton must have played Toronto.

	C	E	H	M	Q	T	V	W
C		✗				✗		
E	✗		✗	✗	✗	✓	✗	✗
H		✗				✗		
M		✗				✗		
Q		✗				✗		
T	✗	✓	✗	✗	✗		✗	✗
V		✗				✗		
W		✗				✗		

Section 1: Activity 1 (continued)

W was chosen by two sportscasters, thus could not have played H, V, Q, or M. Therefore, Winnipeg must have played Calgary.

	C	E	H	M	Q	T	V	W
C		✗	✗	✗	✗	✗	✗	✓
E	✗		✗	✗	✗	✓	✗	✗
H	✗	✗				✗		✗
M	✗	✗				✗		✗
Q	✗	✗				✗		✗
T	✗	✓	✗	✗	✗		✗	✗
V	✗	✗				✗		✗
W	✓	✗	✗	✗	✗	✗	✗	

V was chosen by two sportscasters, thus could not have played H or M. Therefore, Vancouver must have played Quebec City.

From elimination, Halifax must have played Montreal.

	C	E	H	M	Q	T	V	W
C		✗	✗	✗	✗	✗	✗	✓
E	✗		✗	✗	✗	✓	✗	✗
H	✗	✗		✓	✗	✗	✗	✗
M	✗	✗	✓		✗	✗	✗	✗
Q	✗	✗	✗	✗		✗	✓	✗
T	✗	✓	✗	✗	✗		✗	✗
V	✗	✗	✗	✗	✓	✗		✗
W	✓	✗	✗	✗	✗	✗	✗	

Over the weekend, Edmonton played Toronto, Winnipeg played Calgary, Vancouver played Quebec City, and Halifax played Montreal.

Section 1: Activity 2

1. a. Textbook question "Explore: Look for a Pattern," p. 338

Line Positions	Number of Choices
1st	5
1st, 2nd	5×4
1st, 2nd, 3rd	$5 \times 4 \times 3$
1st, 2nd, 3rd, 4th	$5 \times 4 \times 3 \times 2$
1st, 2nd, 3rd, 4th, 5th	$5 \times 4 \times 3 \times 2 \times 1$

b. Textbook questions 1 to 6 of "Inquire," p. 338

1. If there are n people being considered, there are n choices for the 1st position, $n - 1$ choices for the second position, and so on. As the number of positions increases, the number of choices is the product of k numbers, where k is the same as the number of positions. Each number in the product decreases by 1.

2. The number of ways the first 4 people can be arranged is $11 \times 10 \times 9 \times 8 = 7920$.

The number of ways all 11 people can be arranged is
$11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 39\,916\,800$.

3. The number of ways of arranging n people is $n \times (n-1) \times (n-2) \times \ldots \times 3 \times 2 \times 1$.

There are n people for the first choice. Once one person has been placed, there is one less person to arrange; so, there are $n - 1$ people for the second position. This continues until you have only one person remaining for the last position.

4. The number of ways of selecting and arranging r objects from a set of n objects is similar to arranging all n objects. However, because only r objects are being arranged, there are only r numbers in the product.

$\therefore$ Number of ways of selecting and arranging the r objects $= \underbrace{n \times (n-1) \times \ldots \times (n-r+1)}_{r \text{ numbers in the product}}$

For example, if 4 objects are selected and arranged from a set of 10 objects, the number of arrangements is $10 \times 9 \times 8 \times 7$. Notice that the last number, 7, is equal to $10 - 4 + 1$.

Section 1: Activity 2 (continued)

5. The number of distinguishable arrangements decreases because you cannot tell the twins apart. If twin A was in the first position and twin B in the last position, it would appear to be the same arrangement as if twin B was in the first position and twin A was in the last position. The number of distinguishable arrangements is one-half the total number of arrangements.

6. A diagram can be used to detemine the number of different appearing arrangements of the Baca family, assuming the identical triplets are dressed alike. Let M and F represent the mother and father, and let T represent each of the triplets.

   The family could have the following different appearing arrangements:

mother in 1st or father in 1st	MFTTT, MTFTT, MTTFT, MTTTF FMTTT, FTMTT, FTTMT, FTTTM
mother in 2nd or father in 2nd	TMFTT, TMTFT, TMTTF TFMTT, TFTMT, TFTTM
mother in 3rd or father in 3rd	TTMFT, TTMTF, TTFMT, TTFTM
mother in 4th or father in 4th	TTTMF, TTTFM

   There are 20 distinguishable ways the family can be arranged in a line.

2. Textbook questions 1, 3, 5, 11, 12, and 13 of "Practice," p. 342

1. $5 \times 4 \times 3 \times 2 \times 1 = 5!$

3. $8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 8!$

5. $4! = 4 \times 3 \times 2 \times 1$

11. $\dfrac{8!}{4!3!} = \dfrac{8\times7\times6\times5\times\cancel{4}\times\cancel{3}\times\cancel{2}\times\cancel{1}}{(\cancel{4}\times\cancel{3}\times\cancel{2}\times\cancel{1})(3\times2\times1)}$

$= \dfrac{8\times7\times\cancel{6}\times5}{\cancel{3\times2\times1}}$

$= 8\times7\times5$

12. $\dfrac{14!}{9!3!2!} = \dfrac{14\times13\times12\times11\times10\times\cancel{9}\times\cancel{8}\times\cancel{7}\times\cancel{6}\times\cancel{5}\times\cancel{4}\times\cancel{3}\times\cancel{2}\times\cancel{1}}{(\cancel{9}\times\cancel{8}\times\cancel{7}\times\cancel{6}\times\cancel{5}\times\cancel{4}\times\cancel{3}\times\cancel{2}\times\cancel{1})(3\times2\times1)(2\times1)}$

$= \dfrac{14\times13\times\cancel{12}\times11\times10}{\cancel{(3\times2\times1)(2\times1)}}$

$= 14\times13\times11\times10$

13. $\dfrac{10!}{5!2!} = \dfrac{10\times9\times8\times7\times6\times\cancel{5!}}{\cancel{5!}2!}$

$= \dfrac{10\times9\times8\times7\times\overset{3}{\cancel{6}}}{\cancel{2}\times1}$

$= 10\times9\times8\times7\times3$

3. There are four consonants: C, P, L, and S. There are three vowels: O, U, and E. If the consonants, *c*, and vowels, *v*, must alternate, each arrangement is of the form *cvcvcvc*.

Arrange the consonants first.

4 *v* **3** *v* **2** *v* **1**

Next, arrange the vowels.

4 **3** 3 **2** 2 **1** 1

$\therefore$ Number of arrangements $= 4!\times3!$

$= 24\times6$

$= 144$

There are 144 possible letter arrangements that can be formed.

Section 1: Activity 2 (continued)

4. If numbers formed are even, they must end in a 2 or a 4.

Therefore, there are only 2 choices for the last digit.

$$\_\ \_\ \_\ \underline{\mathbf{2}}$$

The remaining three digits can be positioned in any of 3! arrangements.

$$\underline{\mathbf{3}}\ \underline{\mathbf{2}}\ \underline{\mathbf{1}}\ \underline{2}$$

Therefore, the number of even, 4-digit numbers that can be formed is

$$\begin{aligned} 3! \times 2 &= 6 \times 2 \\ &= 12 \end{aligned}$$

5. If the math books are kept together, treat them as a single book.

There are the group of math books and five other books to arrange. These six elements can be arranged in 6! ways.

Next, arrange the four math books among themselves. They can be arranged in 4! ways.

Using the Fundamental Counting Principle, the total number of arrangements possible is

$$\begin{aligned} 6! \times 4! &= 720 \times 24 \\ &= 17\,280 \end{aligned}$$

6. Use an indirect approach.

First, determine the number of arrangements without restriction. Because there are six letters, 6! = 720 arrangements are possible.

Next, determine the number of arrangements with A and E together. Treat them as a single letter, reducing the number of letters from six to five. Therefore, there are 5! = 120 arrangements of five letters. Because A and E can be interchanged, the total number of arrangements with A and E together is 5!2! = 240. Therefore, the number of arrangements possible if the vowels A and E are not adjacent is

$$\begin{aligned} 6! - 5!2! &= 720 - 240 \\ &= 480 \end{aligned}$$

7. Suppose the three math books were first, the four science books second, and the five history books in the last position. There would be 3!4!5! arrangements possible.

But the three subjects can be arranged in 3! ways. Therefore, the number of arrangements possible is $3!4!5! \times 3! = 103\,680$.

8. a. In the word SASKATOON, there are two Ss, two As, and two Os. There are nine letters altogether.

$$\therefore \text{ Number of distinguishable arrangements} = \frac{9!}{2!2!2!}$$
$$= 45\,360$$

b. In the word MISSISSIPPI, there are four Is, four Ss, and two Ps. There are 11 letters altogether.

$$\therefore \text{ Number of distinguishable arrangements} = \frac{11!}{4!4!2!}$$
$$= 34\,650$$

9. There are two 2s and two 4s. There are five digits altogether.

$$\therefore \text{ Number of different numbers possible} = \frac{5!}{2!2!}$$
$$= 30$$

10. a. Textbook questions 8, 9, 10, 16, 18, and 20 of "Practice," p. 342

8. $${}_6P_4 = \frac{6!}{(6-4)!}$$
$$= \frac{6!}{2!}$$
$$= \frac{6 \times 5 \times 4 \times 3 \times \cancel{2!}}{\cancel{2!}}$$
$$= 6 \times 5 \times 4 \times 3$$
$$= 360$$

9. $${}_{11}P_3 = \frac{11!}{(11-3)!}$$
$$= \frac{11!}{8!}$$
$$= \frac{11 \times 10 \times 9 \times \cancel{8!}}{\cancel{8!}}$$
$$= 11 \times 10 \times 9$$
$$= 990$$

Section 1: Activity 2 (continued)

10. $${}_9P_5 = \frac{9!}{(9-5)!}$$
$$= \frac{9!}{4!}$$
$$= \frac{9\times8\times7\times6\times5\times \cancel{4!}}{\cancel{4!}}$$
$$= 9\times8\times7\times6\times5$$
$$= 15\,120$$

16. $$\underbrace{17\times16\times15\times14\times13}_{5 \text{ numbers}} = {}_{17}P_5$$

18. $$\underbrace{8\times7\times6\times5\times4}_{5 \text{ numbers}} = {}_8P_5$$

20. $$\underbrace{88\times87}_{2 \text{ numbers}} = {}_{88}P_2$$

b. Textbook questions 21 to 25.a. of "Applications and Problem Solving," p. 342

21. a. $3\times3\times3 = 27$

There are 27 three-letter codes if letters can be repeated.

b. $3\times2\times1 = 6$

There are 6 three-letter codes if letters cannot be repeated.

22. a. $8\times7\times6\times5\times4\times3\times2\times1 = 8!$
$= 40\,320$

There are 40 320 ways to arrange the chess club members.

b. $${}_8P_3 = \frac{8!}{(8-3!)}$$
$$= \frac{8!}{5!}$$
$$= \frac{8\times7\times6\times\cancel{5!}}{\cancel{5!}}$$
$$= 336$$

There are 336 ways the club can choose a president, vice president, and treasurer.

c. Answers may vary. A sample answer is given.

A chess club has eight members. The club has been asked to have four members speak at a dinner. These four members have been asked to speak about their club's fundraising, membership, future goals, and meeting schedule. Each speaker provides information on only one topic and no topic is repeated. In how many ways can each topic be assigned to the chess club members?

23. a. There are 24! arrangements of a deck of euchre cards.

b. The five cards in a hand can be arranged in 5! = 120 ways.

c. You are laying out 5 cards in a row using the 24 cards. There are 24 cards for the first spot, 23 cards for the second spot, 22 cards for the third spot, 21 cards for the fourth spot, and 20 cards for the fifth spot.

$$\begin{aligned}\therefore {}_{24}P_5 &= \frac{24!}{(24-5)!} \\ &= \frac{24!}{19!} \\ &= \frac{24\times 23\times 22\times 21\times 20\times \cancel{19!}}{\cancel{19!}} \\ &= 5\,100\,480\end{aligned}$$

From a euchre deck, 5 100 480 permutations could be created.

24. a. The number of different signals using all four flags is 4! = 24.

b. The number of different signals using three of the four flags is

$$\begin{aligned}{}_4P_3 &= \frac{4!}{(4-3)!} \\ &= \frac{4!}{1!} \\ &= 4! \\ &= 24\end{aligned}$$

Section 1: Activity 2 (continued)

c. The number of different signals using two of the four flags is

$$\begin{aligned} {}_4P_2 &= \frac{4!}{(4-2)!} \\ &= \frac{4!}{2!} \\ &= \frac{4\times 3\times \cancel{2!}}{\cancel{2!}} \\ &= 12 \end{aligned}$$

The number of different signals using one of the four flags is

$$\begin{aligned} {}_4P_1 &= \frac{4!}{(4-1)!} \\ &= \frac{4!}{3!} \\ &= 4 \end{aligned}$$

Therefore, to determine the total number of ways of using any number of flags, sum the total of using four flags, three flags, two flags, or one flag.

$$\begin{aligned} \text{Number of ways} &= {}_4P_4 + {}_4P_3 + {}_4P_2 + {}_4P_1 \\ &= 24 + 24 + 12 + 4 \\ &= 64 \end{aligned}$$

d. The total number of flags is n.

$$\begin{aligned} \therefore\ {}_nP_2 &= 90 \\ \frac{n!}{(n-2)!} &= 90 \\ \frac{n\times(n-1)\times\cancel{(n-2)!}}{\cancel{(n-2)!}} &= 90 \\ n(n-1) &= 90 \\ n^2 - n &= 90 \\ n^2 - n - 90 &= 0 \\ (n-10)(n+9) &= 0 \end{aligned}$$

$$\begin{aligned} \therefore\ n-10 &= 0 \quad \text{or} \quad n+9 = 0 \\ n &= 10 \qquad\qquad n = -9 \end{aligned}$$

Because the number of flags is non-negative, 10 flags are needed.

25. a. Marion has to walk five blocks east and three blocks north. The number of pathways is the same as the number of ways of arranging five Es and 3 Ns.

There are a total of eight letters.

$$\therefore \text{ Number of pathways} = \frac{8!}{5!3!}$$
$$= 56$$

Marion can take 56 different routes.

11. a.
$${}_nP_2 = 72$$
$$\frac{n!}{(n-2)!} = 72$$
$$\frac{n(n-1)\cancel{(n-2)!}}{\cancel{(n-2)!}} = 72$$
$$n(n-1) = 72$$
$$n^2 - n = 72$$
$$n^2 - n - 72 = 0$$
$$(n-9)(n+8) = 0$$

$\therefore n-9=0$ or $n+8=0$

$n=9$ $\quad$ $n=-8$

Because $n \geq 2$, $n = 9$.

Section 1: Activity 2 (continued)

b.

$$
\begin{aligned}
{}_nP_3 &= 42n \\
\frac{n!}{(n-3)!} &= 42n \\
\frac{n(n-1)(n-2)\cancel{(n-3)!}}{\cancel{(n-3)!}} &= 42n \\
n(n-1)(n-2) &= 42n \\
n^3 - 3n^2 + 2n &= 42n \\
n^3 - 3n^2 - 40n &= 0 \\
n\left(n^2 - 3n - 40\right) &= 0 \\
n(n-8)(n+5) &= 0
\end{aligned}
$$

$\therefore n = 0$ or $n - 8 = 0$ or $n + 5 = 0$

$n = 8$ $\quad$ $n = -5$

Because $n \geq 3$, $n = 8$.

12. Answers may vary. A sample answer is given.

A coach is always concerned about the order of wins, losses, and ties because they indicate trends. If the wins were all at the beginning of the season and the losses were all at the end of the season, this trend would not look good for the playoffs!

13. Textbook question 27 of "Applications and Problem Solving," p. 342

27. a. The total number of games played is $15 + 4 + 2 = 21$ games. This is a permutation with repetitions.

$$
\begin{aligned}
\therefore \text{Number of orders} &= \frac{21!}{15!4!2!} \\
&= 813\,960
\end{aligned}
$$

The record could have occurred 813 960 ways.

b. If the team began with 5 wins, you now have to arrange $21 - 5 = 16$ games. The 16 games consist of 10 wins, 4 losses, and 2 ties. This is a permutation with repetitions.

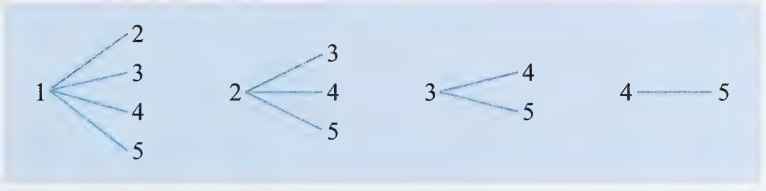
$$\therefore \text{ Number of orders} = \frac{16!}{10!4!2!}$$
$$= 120\,120$$

With five straight wins, the record could have occurred 120 120 ways.

Section 1: Activity 3

1. a. Textbook question "Explore: Look for a Pattern," p. 345

The two numbers must be chosen from the set {1, 2, 3, 4, 5}. Draw a tree diagram showing all the possible two-number combinations.



All the possible two-number combinations are

{1, 2}, {1, 3}, {1, 4}, {1, 5}, {2, 3}, {2, 4}, {2, 5}, {3, 4}, {3, 5}, and {4, 5}

b. Textbook questions 1 to 4 of "Inquire," p. 345

1. The tree diagram is different from the ones used in the first section of the chapter because order is not important. For example, both {1, 2} and {2, 1} are considered the same choice and, thus, should not be repeated.

The Fundamental Counting Principle cannot be used to count the number of combinations because order is involved when the Fundamental Counting Principle is applied.

There are 10 two-number combinations for the 2/5 lottery.

Section 1: Activity 3 (continued)

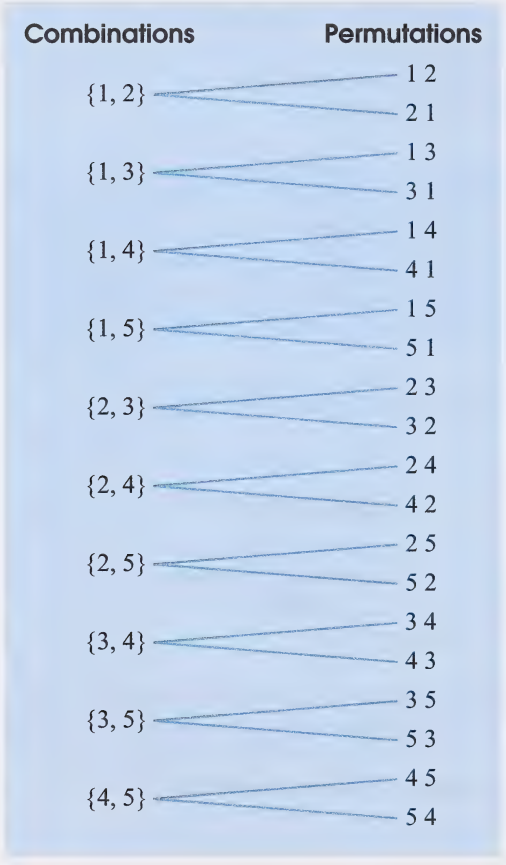
2. The number of permutations of two numbers from {1, 2, 3, 4, 5} with no repetitions is

picking the 1st number

$${}_5P_2 = 5 \times 4$$

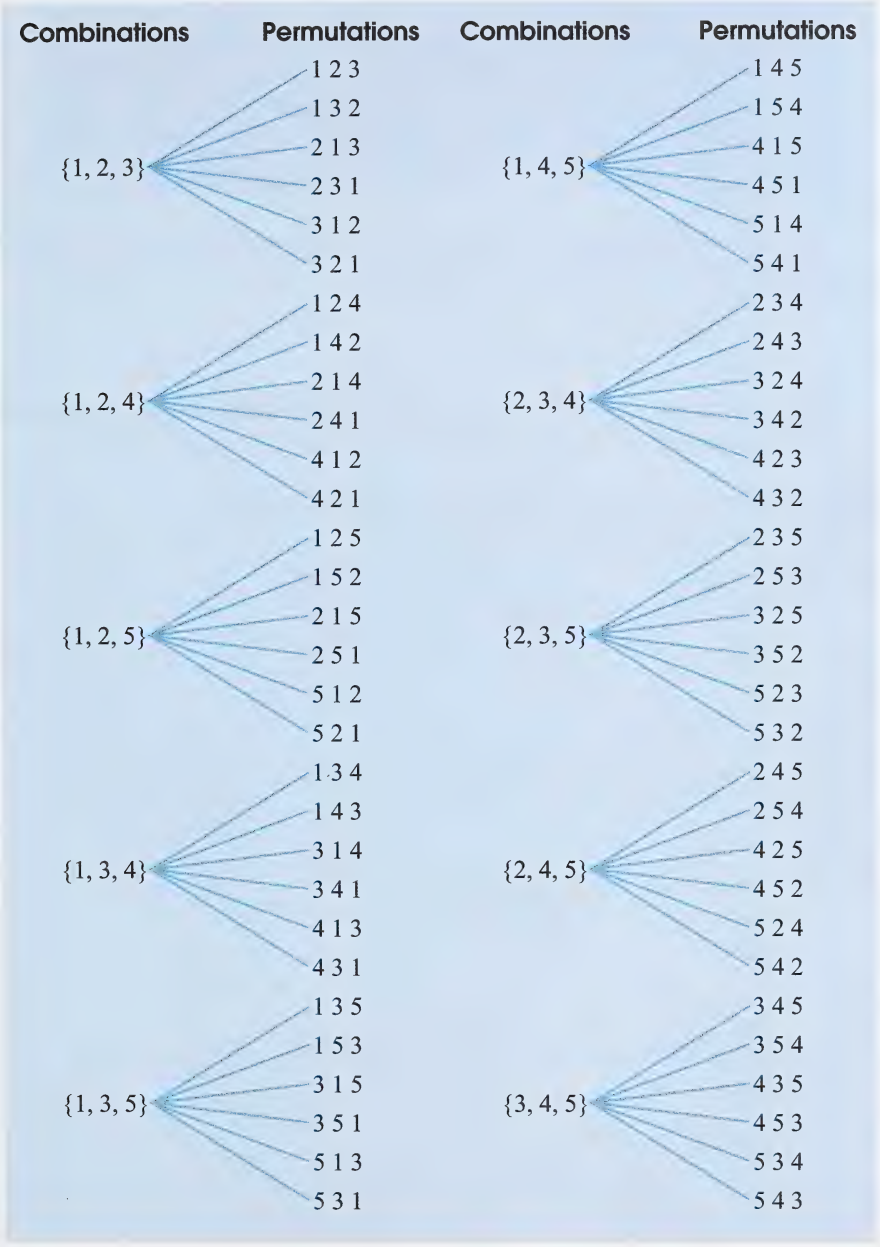
picking the 2nd number after one number has been used

Each of the combinations in question 1.a. can be arranged in two ways.



The second list has double the number of pairs compared to the first list.

3. Assume the three numbers are chosen from the set {1, 2, 3, 4, 5}.



There are 3! or 6 times as many permutations as combinations.

Order is important with permutations, but not with combinations.

Section 1: Activity 3 (continued)

4.

Lottery	Number of Permutations	Number of Combinations
2/5	20	10
3/5	60	10

There are always more permutations than combinations, and the number of permutations is a multiple of the number of combinations.

e.g., $\frac{20}{2}=10$ and $\frac{60}{6}=10$

Notice that 2 = 2! and 6 = 3!. Therefore, the number of combinations is the number of permutations of k objects from n objects divided by $k!$.

2. a. Textbook questions 2, 3, 6, 9, 10, 11, and 14 of "Practice," p. 348

2. $${}_5C_2=\frac{5!}{(5-2)!2!}$$
$$=\frac{5!}{3!2!}$$
$$=\frac{5\times\overset{2}{\cancel{4}}\times\overset{1}{\cancel{3!}}}{\underset{1}{\cancel{3!}}\times\underset{1}{\cancel{2}}\times 1}$$
$$=5\times 2$$

Recall: ${}_nC_r=\frac{n!}{(n-r)!r!}$

3. $${}_6C_4=\frac{6!}{(6-4)!4!}$$
$$=\frac{6!}{2!4!}$$
$$=\frac{\overset{3}{\cancel{6}}\times 5\times\overset{1}{\cancel{4!}}}{\underset{1}{\cancel{2}}\times 1\times\underset{1}{\cancel{4!}}}$$
$$=3\times 5$$

6. $${}_{29}C_5 = \frac{29!}{(29-5)!5!}$$

$$= \frac{29!}{24!5!}$$

$$= \frac{29 \times \overset{7}{\cancel{28}} \times \overset{9}{\cancel{27}} \times \overset{13}{\cancel{26}} \times \overset{5}{\cancel{25}} \times \overset{1}{\cancel{24!}}}{\underset{1}{\cancel{24!}} \times \underset{1}{\cancel{5}} \times \underset{1}{\cancel{4}} \times \underset{1}{\cancel{3}} \times \underset{1}{\cancel{2}} \times 1}$$

$$= 29 \times 7 \times 9 \times 13 \times 5$$

9. $$\frac{7 \times 6 \times 5}{3 \times 2 \times 1} = \frac{7 \times 6 \times 5 \times 4!}{3!4!}$$ ← Multiply the numerator and denominator by 4!.

$$= \frac{7!}{3!4!}$$

$$= \frac{7!}{(7-4)!4!} \text{ or } \frac{7!}{(7-3)!3!}$$

$$= {}_7C_4 \text{ or } {}_7C_3$$

10. $$17 \times 16 \times 15 \times 14 \times 13 = \frac{17 \times 16 \times 15 \times 14 \times 13 \times 12!}{12!}$$ ← Multiply the numerator and denominator by 12!.

$$= \frac{17!}{12!}$$

$$= \frac{17!}{(17-5)!}$$

$$= {}_{17}P_5$$

11. $$\frac{17 \times 16 \times 15 \times 14 \times 13}{5 \times 4 \times 3 \times 2 \times 1} = \frac{17 \times 16 \times 15 \times 14 \times 13 \times 12!}{5!12!}$$ ← Multiply the numerator and denominator by 12!.

$$= \frac{17!}{5!12!}$$

$$= \frac{17!}{(17-12)!12!} \text{ or } \frac{17!}{(17-5)!5!}$$

$$= {}_{17}C_{12} \text{ or } {}_{17}C_5$$

14. $$\frac{74 \times 73 \times 72 \times 71}{4 \times 3 \times 2 \times 1} = \frac{74 \times 73 \times 72 \times 71 \times 70!}{4!70!}$$ ← Multiply the numerator and denominator by 70!.

$$= \frac{74!}{4!70!}$$

$$= \frac{74!}{(74-70)!70!} \text{ or } \frac{74!}{(74-4)!4!}$$

$$= {}_{74}C_{70} \text{ or } {}_{74}C_4$$

Section 1: Activity 3 (continued)

b. Textbook question 15 of "Applications and Problem Solving," p. 348

15. a. There are six colours: red, white, blue, green, yellow, and black.

$n = 6$ and $r = 2$

$$\therefore {}_nC_r = \frac{n!}{(n-r)!r!}$$

$${}_6C_2 = \frac{6!}{(6-2)!2!}$$

$$= \frac{6!}{4!2!}$$

$$= \frac{\overset{3}{\cancel{6}} \times 5 \times \overset{1}{\cancel{4!}}}{\underset{1}{\cancel{4!}} \times \underset{1}{\cancel{2}} \times 1}$$

$$= 15$$

There are 15 two-colour combinations.

b. $n = 6$ and $r = 3$

$$\therefore {}_nC_r = \frac{n!}{(n-r)!r!}$$

$${}_6C_3 = \frac{6!}{(6-3)!3!}$$

$$= \frac{6!}{3!3!}$$

$$= \frac{\cancel{6} \times 5 \times 4 \times \cancel{3!}}{\cancel{3!}\,\cancel{(3 \times 2 \times 1)}}$$

$$= 20$$

There are 20 three-colour combinations.

c. Answers will vary. A sample answer is given.

Some countries that have flags with two-colour combinations are Canada, Austria, Indonesia, Latvia, Monaco, Poland, and Ukraine.

Some countries that have flags with three-colour combinations are the United States, France, Ireland, Italy, and Russia.

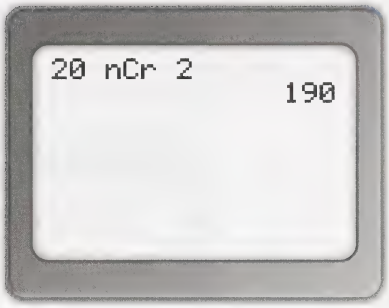
3. Number of diagonals $= {}_nC_2 - n$

$= {}_{20}C_2 - 20$

$= 190 - 20$

$= 170$

A regular polygon with 20 sides has 170 diagonals.



4. Triangles are formed by joining 3 vertices chosen from the 20 vertices in the polygon.

$\therefore$ Number of triangles $= {}_{20}C_3$

$= 1140$

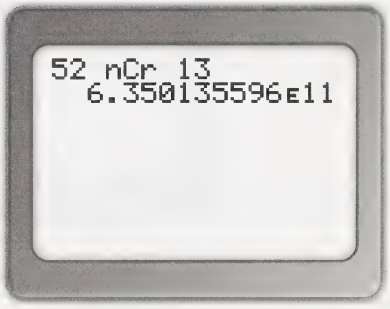
A total of 1140 triangles can be formed.

5. Textbook questions 16, 17, 18, 19.b., 19.e., 19.f., 20, 21, and 23 of "Applications and Problem Solving," pp. 348 and 349

16. a. A deck of cards consists of 52 cards and each hand consists of 13 cards.

$\therefore n = 52$ and $r = 13$

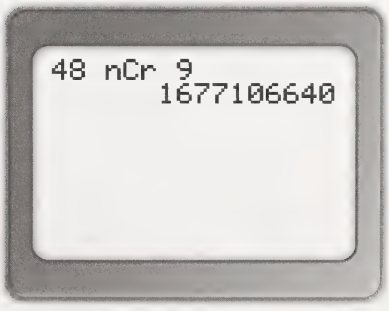
There are approximately 6.35×10^{11} different hands.



b. If a bridge hand has 4 aces, then only 9 cards (13 – 4) need to be chosen from 48 cards (52 cards – 4 aces).

$\therefore n = 48$ and $r = 9$

There are 1 677 106 640 different hands with four aces.



Section 1: Activity 3 (continued)

c. $n = 48$ and $r = 13$

There are approximately 1.93×10^{11} different hands with no aces.

17. a. Total number of family members $= 2 + 11 + 115 + 529 + 20$
$= 677$

$\therefore n = 677$ and $r = 20$

There are approximately 1.27×10^{38} different combinations of family members that could be at the head table.

b. $n = 677 - 529 = 148$ and $r = 20$

There are approximately 2.72×10^{24} different combinations if none of the great-grandchildren sit at the head table.

c. To determine the number of head-table combinations that include at least 1 great-grandchild, subtract the total number of head-table combinations that include no great-grandchildren, ${}_{148}C_{20}$, from the total number of head-table combinations, ${}_{677}C_{20}$.

There are approximately 1.27×10^{38} different combinations if at least 1 great-grandchild sits at the head table.

Section 1: Activity 3 (continued)

d. $n = 677$ and $r = 2$

If every family member shook every other family member's hand, 228 826 handshakes would take place.

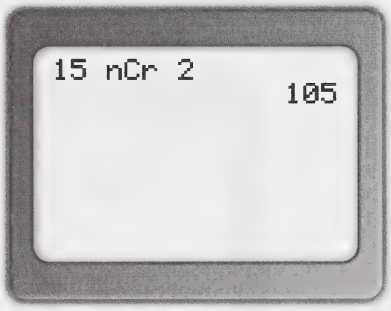
18. a. $n = 15$ and $r = 2$

$$\therefore {}_nC_r = \frac{n!}{(n-r)!r!}$$

$${}_{15}C_2 = \frac{15!}{(15-2)!2!}$$

$$= \frac{15!}{13!2!}$$

$$= 105$$



The company can make 105 different two-topping pizzas.

b. $n = 15$ and $r = 5$

$$\therefore {}_nC_r = \frac{n!}{(n-r)!r!}$$

$${}_{15}C_5 = \frac{15!}{(15-5)!5!}$$

$$= \frac{15!}{10!5!}$$

$$= 3003$$

The company can make 3003 different five-topping pizzas.

c. $n = 14$ and $r = 3$

$$\therefore \ {}_nC_r = \frac{n!}{(n-r)!r!}$$

$${}_{14}C_3 = \frac{14!}{(14-3)!3!}$$

$$= \frac{14!}{11!3!}$$

$$= 364$$

The company can make 364 different three-topping pizzas that do not include anchovies.

d. Out of 15 toppings, 6 are vegetables.

$\therefore n = 9$ and $r = 3$

$$\therefore \ {}_nC_r = \frac{n!}{(n-r)!r!}$$

$${}_9C_3 = \frac{9!}{(9-3)!3!}$$

$$= \frac{9!}{6!3!}$$

$$= 84$$

The company can make 84 different three-topping pizzas that include no vegetable toppings.

e. Total $= {}_{15}C_3 - {}_9C_3$

$$= 455 - 84$$

$$= 371$$

The company can make 371 different three-topping pizzas with at least one vegetable topping.

Section 1: Activity 3 (continued)

19. b.

$$
\begin{aligned}
{}_nC_2 &= 6 \\
\frac{n!}{(n-2)!2!} &= 6 \\
\frac{n(n-1)\cancel{(n-2)!}}{\cancel{(n-2)!}\,2!} &= 6 \\
\frac{n(n-1)}{2\times 1} &= 6 \\
n(n-1) &= 12 \\
n^2 - n &= 12 \\
n^2 - n - 12 &= 0 \\
(n-4)(n+3) &= 0
\end{aligned}
$$

$\therefore\ n-4=0 \quad \text{or} \quad n+3=0$

$n=4 \qquad\qquad n=-3$

Because $n \geq 2$, $n = 4$.

e.

$$
\begin{aligned}
{}_nC_3 &= 3\left({}_nP_2\right) \\
\frac{\cancel{n!}}{(n-3)!3!} &= 3\left(\frac{\cancel{n!}}{(n-2)!}\right) \\
\frac{1}{(n-3)!3!} &= \frac{3}{(n-2)!} \\
\frac{(n-2)!}{(n-3)!} &= 3(3!) \\
\frac{(n-2)\cancel{(n-3)!}}{\cancel{(n-3)!}} &= 3(3\times 2\times 1) \\
n-2 &= 18 \\
n &= 20
\end{aligned}
$$

f.

$$
\begin{aligned}
{}_nP_3 &= 2\left({}_nC_4\right) \\
\frac{n!}{(n-3)!} &= 2\left(\frac{n!}{(n-4)!4!}\right) \\
\frac{1}{(n-3)!} &= \frac{2}{(n-4)!4!} \\
\frac{(n-4)!}{(n-3)!} &= \frac{2}{4\times3\times2\times1} \\
\frac{(n-4)!}{(n-3)(n-4)!} &= \frac{1}{12} \\
\frac{1}{n-3} &= \frac{1}{12} \\
n-3 &= 12 \\
n &= 15
\end{aligned}
$$

20. a. ${}_{10}C_1 = 10$ and ${}_{10}C_9 = 10$

${}_{10}C_2 = 45$ and ${}_{10}C_8 = 45$

${}_{10}C_3 = 120$ and ${}_{10}C_7 = 120$

${}_{10}C_4 = 210$ and ${}_{10}C_6 = 210$

Each pair of combinations has the same value.

For each pair, the values of r equal 10 when added together.

$\therefore {}_{10}C_4 = {}_{10}C_{10-r}$

b. ${}_nC_r = {}_nC_{n-r}$

c.

LS	RS
${}_{50}C_7$	${}_{50}C_{50-7}$
$= \frac{50!}{(50-7)!7!}$	$= {}_{50}C_{43}$
$= \frac{50!}{43!7!}$	$= \frac{50!}{(50-43)!43!}$
	$= \frac{50!}{7!43!}$

LS = RS

Section 1: Activity 3 (continued)

21. a. The 7-person committee is to be chosen from 15 people.

$\therefore n = 15$ and $r = 7$

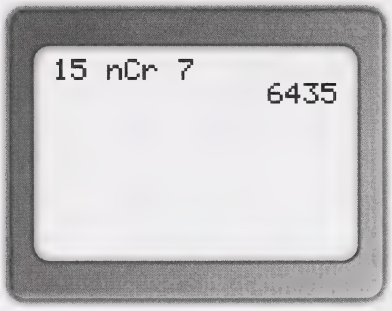
$$\therefore {}_nC_r = \frac{n!}{(n-r)!r!}$$

$${}_{15}C_7 = \frac{15!}{(15-7)!7!}$$

$$= \frac{15!}{8!7!}$$

$$= 6435$$

There are 6435 possible committees that can be formed.



b. If there is no representation from Eastern Canada, then the 7-person committee is chosen from only 10 people.

$\therefore n = 10$ and $r = 7$

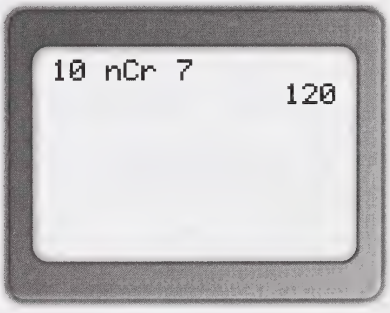
$$\therefore {}_nC_r = \frac{n!}{(n-r)!r!}$$

$${}_{10}C_7 = \frac{10!}{(10-7)!7!}$$

$$= \frac{10!}{3!7!}$$

$$= 120$$

There are 120 possible committees that would have no representation from Eastern Canada.



c. To determine the number of committees that would have at least one representative from each region, subtract the number of committees possible that would have no representation from Eastern Canada, that would have no representation from Central Canada, and that would have no representation from Western Canada from the total number of committees possible.

$\therefore$ Number of committees $= {}_{15}C_7 - ({}_{10}C_7 + {}_{10}C_7 + {}_{10}C_7)$

(no Western; no Eastern; no Central)

$= 6435 - (120 + 120 + 120)$

$= 6435 - 360$

$= 6075$

There are 6075 possible committees that would have at least one representative from each region.

23. a. $n = 52$ and $r = 5$

$$\therefore {}_nC_r = \frac{n!}{(n-r)!r!}$$

$${}_{52}C_5 = \frac{52!}{(52-5)!5!}$$

$$= \frac{52!}{47!5!}$$

$$= 2\,598\,960$$

There are 2 598 960 different five-card hands.

b. There are $52 \div 2 = 26$ red cards in a deck of cards.

$\therefore n = 26$ and $r = 5$

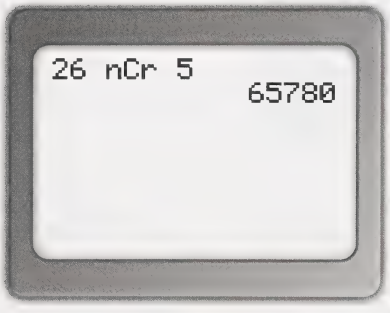
$$\therefore {}_nC_r = \frac{n!}{(n-r)!r!}$$

$${}_{26}C_5 = \frac{26!}{(26-5)!5!}$$

$$= \frac{26!}{21!5!}$$

$$= 65\,780$$

There are 65 780 different five-card hands containing only red cards.



Section 1: Activity 3 (continued)

c. There are 4 suits in a standard deck of cards; therefore, there are $52 \div 4 = 13$ cards in each suit.

$\therefore n = 13$ and $r = 5$

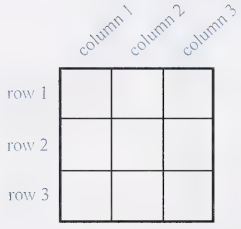
You need to determine the number of different hands containing all cards of the same suit among the 4 different suits.

$$\begin{aligned}\therefore \text{ Number of five-card hands} &= 4\left({}_nC_r\right) \\ &= 4\left({}_{13}C_5\right) \\ &= 4(1287) \\ &= 5148\end{aligned}$$

There are 5148 different five-card hands containing all cards of the same suit.

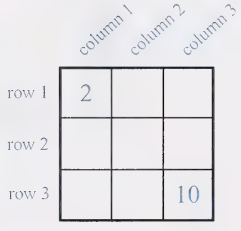
6. Textbook question "NUMBER POWER," p. 349

Use the following diagram to help solve the problem.



The card with the 2 cannot be in row 2, row 3, column 2, or column 3 because there is no card that can be placed above or to the left of it. Therefore, the card with the 2 must be in row 1 column 1.

The card with the 10 cannot be in row 1, row 2, column 1, or column 2 because there is no card that can be placed below or to the right of it. Therefore, the card with the 10 must be in row 3 column 3.



Because there is only one card that can either be above or to the left of the card with the 3, there are two possible locations for the card with the 3: row 1 column 2 or row 2 column 1.

Possibility 1

	column 1	column 2	column 3
row 1	2	3	
row 2			
row 3			10

Possibility 2

	column 1	column 2	column 3
row 1	2		
row 2	3		
row 3			10

There are only 2 cards that can be placed above or to the left of the card with the 4. Therefore, there are two possible locations for the card with the 4 for each of the preceeding possibilities. Consider the following four cases.

Case 1

2	3	4
		10

Case 2

2	3	
4		
		10

Case 3

2		
3		
4		10

Case 4

2	4	
3		
		10

Now, investigate each case separately, and determine every possible location of cards 5 to 9.

Case 1

2	3	4
5	6	7
8	9	10

2	3	4
5	6	8
7	9	10

2	3	4
5	6	9
7	8	10

2	3	4
5	7	8
6	9	10

2	3	4
5	7	9
6	8	10

Note: There is only one location for the card with the 5 in this case: row 2 column 1.

Case 1 has 5 possible arrangements.

Section 1: Activity 3 (continued)

Case 2

2	3	5
4	6	7
8	9	10

2	3	5
4	6	8
7	9	10

2	3	5
4	6	9
7	8	10

2	3	5
4	7	8
6	9	10

2	3	5
4	7	9
6	8	10

2	3	6
4	5	7
8	9	10

2	3	6
4	5	8
7	9	10

2	3	6
4	5	9
7	8	10

2	3	8
4	5	9
6	7	10

2	3	7
4	5	8
6	9	10

2	3	7
4	5	9
6	8	10

2	3	6
4	7	8
5	9	10

2	3	6
4	7	9
5	8	10

2	3	7
4	6	8
5	9	10

2	3	7
4	6	9
5	8	10

2	3	8
4	6	9
5	7	10

Case 2 has 16 possible arrangements.

Case 3

2	5	8
3	6	9
4	7	10

2	5	7
3	6	8
4	9	10

2	5	7
3	6	9
4	8	10

2	5	6
3	7	8
4	9	10

2	5	6
3	7	9
4	8	10

Note: There is only one location for the card with the 5 in this case: row 1 column 2.

Case 3 has 5 possible arrangements.

Case 4

2	4	5
3	6	7
8	9	10

2	4	5
3	6	8
7	9	10

2	4	5
3	6	9
7	8	10

2	4	5
3	7	8
6	9	10

2	4	5
3	7	9
6	8	10

2	4	6
3	5	7
8	9	10

2	4	6
3	5	8
7	9	10

2	4	6
3	5	9
7	8	10

2	4	8
3	5	9
6	7	10

2	4	7
3	5	8
6	9	10

2	4	7
3	5	8
6	9	10

2	4	6
3	7	8
5	9	10

2	4	6
3	7	9
5	8	10

2	4	7
3	6	8
5	9	10

2	4	7
3	6	9
5	8	10

2	4	8
3	6	9
5	7	10

Case 4 has 16 possible arrangements.

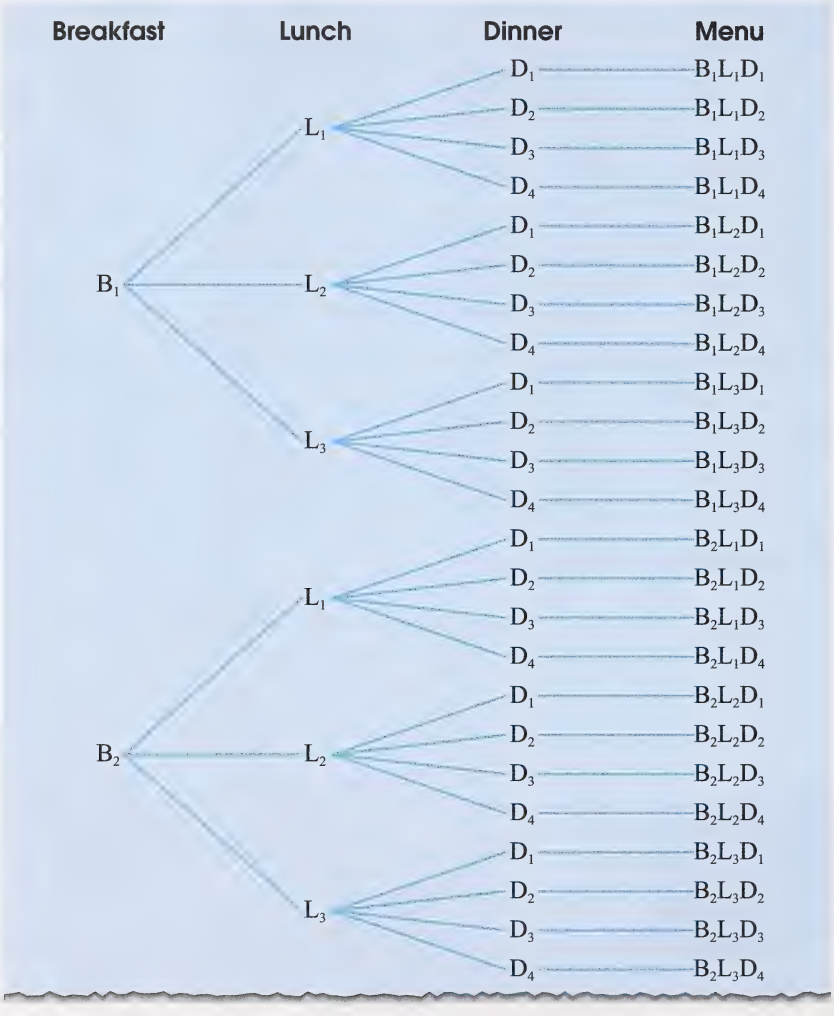
Therefore, there are a total of $5 + 16 + 5 + 16 = 42$ possible arrangements.

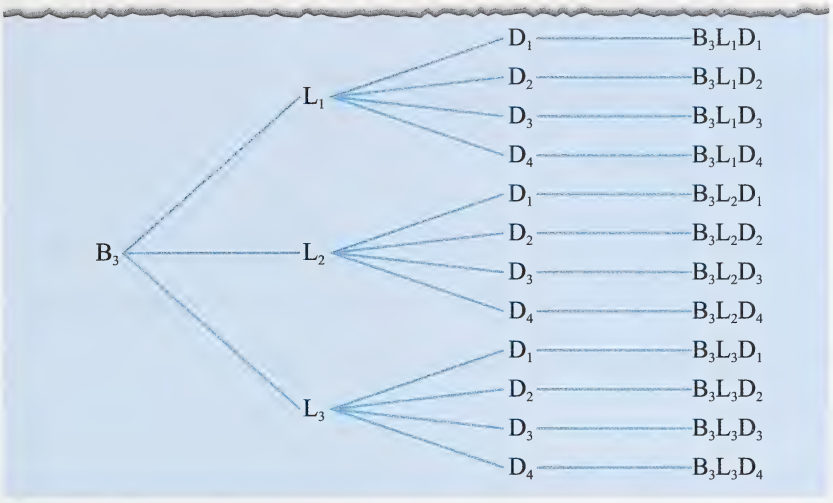
Section 1: Follow-up Activities

Extra Help

Textbook questions 1, 3, 4, 5, 6, and 8 of "Review," p. 360

1. Let B_1, B_2, and B_3 represent the breakfast choices; let L_1, L_2, and L_3 represent the lunch choices; and let D_1, D_2, D_3, and D_4 represent the dinner choices.





There are 36 different daily menus that can be made.

3. a. Because the order matters, this is a permutation.

There are 8 coloured pegs in total, and 5 different coloured pegs make up a secret code. Therefore, $n = 8$ and $r = 5$.

$$\therefore {}_nP_r = \frac{n!}{(n-r)!}$$

$${}_8P_5 = \frac{8!}{(8-5)!}$$

$$= \frac{8!}{3!}$$

$$= \frac{8 \times 7 \times 6 \times 5 \times 4 \times 3!}{3!}$$

$$= 6720$$

There are 6720 different secret codes possible if no two pegs are the same colour.

b. If the colours can be repeated, then there are 8 colours possible for each peg. Therefore, you must use the Fundamental Counting Principle.

$$\text{Total} = 8 \times 8 \times 8 \times 8 \times 8$$

$$= 8^5$$

$$= 32\,768$$

There are 32 768 different secret codes possible if any colour can be repeated.

Extra Help (continued)

c. Allowing blanks is just like adding another possible colour. Therefore, there are 9 possibilities (8 different colours and a blank) for each of the 5 pegs.

$$\therefore \text{Total} = 9 \times 9 \times 9 \times 9 \times 9$$
$$= 9^5$$
$$= 59\,049$$

There are 59 049 different secret codes possible if any colour can be repeated or blanks used.

4. a. Using the letters G S U T Y R A, there are no repetitions.

Method 1

There are 7 letters for the first spot.

$$\underline{7} \times \underline{\quad} \times \underline{\quad}$$

There are 6 letters for the second spot.

$$\underline{7} \times \underline{6} \times \underline{\quad}$$

There are 5 letters for the third spot.

$$\underline{7} \times \underline{6} \times \underline{5}$$

There are $7 \times 6 \times 5 = 210$ possible three-letter words.

Method 2

A "word" is a permutation. There are 7 letters in which 3 are needed. Therefore, $n = 7$ and $r = 3$.

$$\therefore {}_nP_r = \frac{n!}{(n-r)!}$$
$${}_7P_3 = \frac{7!}{(7-3)!}$$
$$= \frac{7!}{4!}$$
$$= \frac{7 \times 6 \times 5 \times \cancel{4!}}{\cancel{4!}}$$
$$= 210$$

There are 210 possible three-letter words.

b. Using the letters F I B W O D L, there are no repetitions.

Method 1

There are 7 letters for the first spot.

$\_7\_ \times \_\_ \times \_\_ \times \_\_$

There are 6 letters for the second spot.

$\_7\_ \times \_6\_ \times \_\_ \times \_\_$

There are 5 letters for the third spot.

$\_7\_ \times \_6\_ \times \_5\_ \times \_\_$

There are 4 letters for the fourth spot.

$\_7\_ \times \_6\_ \times \_5\_ \times \_4\_$

There are $7 \times 6 \times 5 \times 4 = 840$ different four-letter words.

Method 2

You are arranging 4 letters from 7 letters where order matters; therefore, $n = 7$ and $r = 4$.

$$\therefore {}_nP_r = \frac{n!}{(n-r)!}$$

$${}_7P_4 = \frac{7!}{(7-4)!}$$

$$= \frac{7!}{3!}$$

$$= \frac{7 \times 6 \times 5 \times 4 \times \cancel{3!}}{\cancel{3!}}$$

$$= 840$$

There are 840 different four-letter words.

c. Using the letters S S A N T T E, notice that there are two Ss and two Ts. Therefore, you must use the formula for permutations of n objects when some are alike.

$$\frac{n!}{a!b!c!\ldots} = \frac{7!}{2!2!} \quad \leftarrow \text{There are 7 letters altogether.} \quad \leftarrow \text{There are 2 Ss and 2 Ts.}$$

$$= 1260$$

There are 1260 different words possible when you are trying to use all of the tiles.

Extra Help (continued)

5. a. If 7 people line up in a row, there are 7! = 5040 ways to arrange them.

b. Method 1

If the students line up in two rows, with three students in the front and four in the back, the number of ways of arranging these students is exactly the same as the number of ways of arranging them in one row.

Imagine the back row and front row joined to make one row.

Therefore, there are 7! = 5040 ways to arrange the student council for the photograph.

Method 2

Imagine the lines represent the places where the students will stand.

Back Row _ _ _ _
Front Row _ _ _

For the first spot in the back row, there are 7 people to choose from. Therefore, there are only 6 people to choose from for the second spot in the back row, 5 people to choose from for the third spot in the back row, and 4 people to choose from for the last spot in the back row.

Back Row 7 6 5 4
Front Row _ _ _

For the first spot in the front row, there are only 3 people to choose from. Therefore, there are only 2 people to choose from for the second spot in the front row, leaving the last person to fill in the last spot.

Back Row 7 6 5 4
Front Row 3 2 1

Using the Fundamental Counting Principle, the number of ways of arranging the students is

$$\begin{aligned} 7\times6\times5\times4\times3\times2\times1 &= 7! \\ &= 5040 \end{aligned}$$

6. a. Because 4 students are chosen from a class of 28 students and order does not matter, the answer is a combination.

$\therefore n = 28$ and $r = 4$

$$\therefore {}_nC_r = \frac{n!}{(n-r)!r!}$$

$${}_{28}C_4 = \frac{28!}{(28-4)!4!}$$

$$= \frac{28!}{24!4!}$$

$$= 20\,475$$

There are 20 475 different volunteer groups possible.

b. If Miye, one of the students in the class, must be in the group, then 3 students need to be chosen from the remaining 27 students in the class.

$\therefore n = 27$ and $r = 3$

$$\therefore {}_nC_r = \frac{n!}{(n-r)!r!}$$

$${}_{27}C_3 = \frac{27!}{(27-3)!3!}$$

$$= \frac{27!}{24!3!}$$

$$= 2925$$

There are 2925 different volunteer groups possible if Miye must be one of them.

8. a. Assuming the friends never play squash twice on the same day, they will play squash on 3 of the 7 days of the week.

$\therefore n = 7$ and $r = 3$

$$\therefore {}_nC_r = \frac{n!}{(n-r)!r!}$$

$${}_7C_3 = \frac{7!}{(7-3)!3!}$$

$$= \frac{7!}{4!3!}$$

$$= 35$$

There are 35 ways the two friends can select the days on which to play squash.

Extra Help (continued)

b. To play 4 times a week, the two friends are choosing 4 days of the 7 days a week.

$\therefore n = 7$ and $r = 4$

$$\therefore {}_nC_r = \frac{n!}{(n-r)!}$$

$${}_7C_4 = \frac{7!}{(7-4)!4!}$$

$$= \frac{7!}{3!4!}$$

$$= 35$$

There are 35 ways the two friends can select the days on which to play squash.

Enrichment

1. Let A, B, C, and D represent the four people.

Group 1	Group 2
AB	CD
AC	BD
AD	BC

Therefore, there are 3 different ways four people can be divided into two groups of two.

2. The number of ways 8 people can be divided into 2 groups of 4 is

$$\frac{{}_8C_4}{2} = \frac{8!}{(8-4)!4!} \times \frac{1}{2}$$

$$= \frac{8!}{4!4!2}$$

$$= 35$$

3. The number of ways $2n$ people can be divided into 2 groups of n people is $\frac{{}_{2n}C_n}{2}$.

4. a. Let A, B, C, D, E, and F represent the six people.

Group 1	Group 2	Group 3
AB	CD	EF
AB	CE	DF
AB	CF	DE
AC	BD	EF
AC	BE	DF
AC	BF	DE
AD	BC	EF
AD	BE	CF
AD	BF	CE
AE	BC	DF
AE	BD	CF
AE	BF	CD
AF	BC	DE
AF	BD	CE
AF	BE	CD

There are 15 ways six people can be divided into three groups of two.

b. Number of ways $= \dfrac{{}_9C_3 \times {}_6C_3 \times {}_3C_3}{3!}$

$$= \frac{\frac{9!}{(9-3)!3!} \times \frac{6!}{(6-3)!3!} \times \frac{3!}{(3-3)!3!}}{3!}$$

$$= \frac{\frac{9!}{6!3!} \times \frac{6!}{3!3!} \times \frac{3!}{0!3!}}{3!}$$

$$= \frac{\overset{14}{\cancel{84}} \times 20 \times 1}{\underset{1}{\cancel{6}}}$$

$$= 280$$

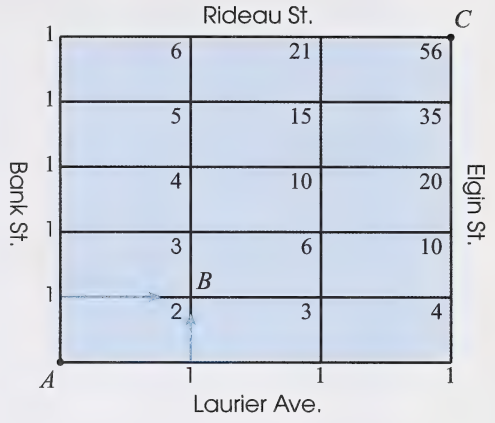
You must divide by 3! because you are not distinguishing among the three groups.

There are 280 ways nine people can be divided into three equal groups.

Section 2: Activity 1

1. a. Textbook question "Explore: Look for a Pattern," p. 350

The number of routes to each intersection are shown on the map given on the right.



b. Textbook questions 1 to 4 of "Inquire," pp. 350 and 351

1. The number of different routes from A to C is 56. **Note:** The number of different routes could also be determined by representing each block that is walked North with N and each block that is walked East with E. The letters that have to be arranged are N N N N N E E E. The number of ways to arrange these letters is $\frac{8!}{5!3!} = 56$.

2. The number of routes for each point of intersection is at a vertex of the "triangle." To obtain the value at the vertex, sum the number of routes at the other two vertices of the "triangle."

3.

$n = 0$ $\quad {}_0C_0 = 1$

$n = 1$ $\quad {}_1C_0 = 1 \quad {}_1C_1 = 1$

$n = 2$ $\quad {}_2C_0 = 1 \quad {}_2C_1 = 2 \quad {}_2C_2 = 1$

$n = 3$ $\quad {}_3C_0 = 1 \quad {}_3C_1 = 3 \quad {}_3C_2 = 3 \quad {}_3C_3 = 1$

$\boldsymbol{n = 4}$ $\quad \boldsymbol{{}_4C_0 = 1 \quad {}_4C_1 = 4 \quad {}_4C_2 = 6 \quad {}_4C_3 = 4 \quad {}_4C_4 = 1}$

$\boldsymbol{n = 5}$ $\quad \boldsymbol{{}_5C_0 = 1 \quad {}_5C_1 = 5 \quad {}_5C_2 = 10 \quad {}_5C_3 = 10 \quad {}_5C_4 = 5 \quad {}_5C_5 = 1}$

4.

1

1 1

1 2 1

1 3 3 1

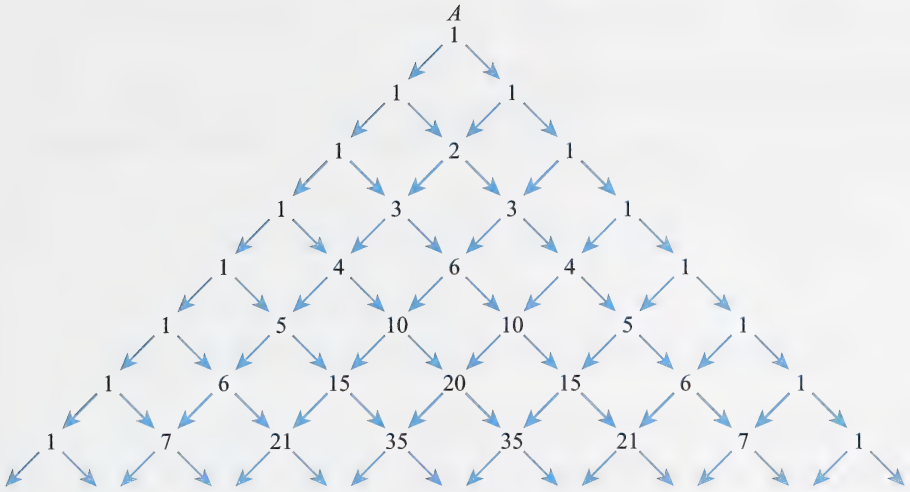
1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 **7 21 35 35 21 7 1**

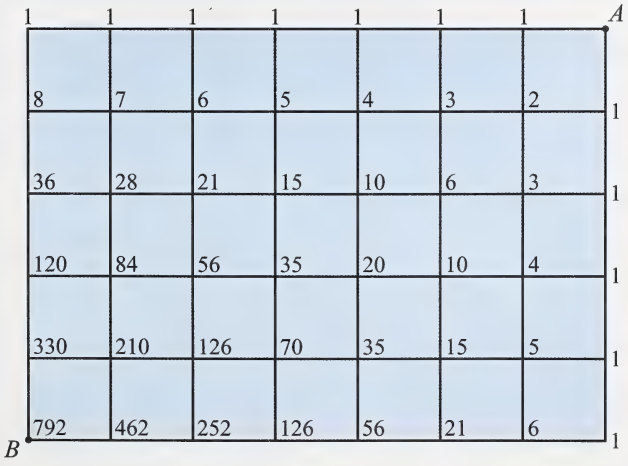
2.



3. Logically, it does not matter whether you work your way downward to the left and then to the right or downward to the right and then to the left. Because of the symmetry of the shape of this triangular array, the entries in the rows are the same whether they are read forward or backward. You should be able to flip the diagram in question 2 over and get a resulting figure identical to the original.

4. Method 1: Using Pascal's Triangle

Draw the diagram with *A* in the northeast corner and *B* in the southwest corner. Then apply the techniques of Pascal's triangle.



There are 792 different paths from the northeast corner to the southwest corner.

Section 2: Activity 1 (continued)

Method 2: Using Distinguishable Arrangements

To go from A to B, you need to go 7 blocks west and 5 blocks south. This corresponds to the number of arrangements of W W W W W W W S S S S S.

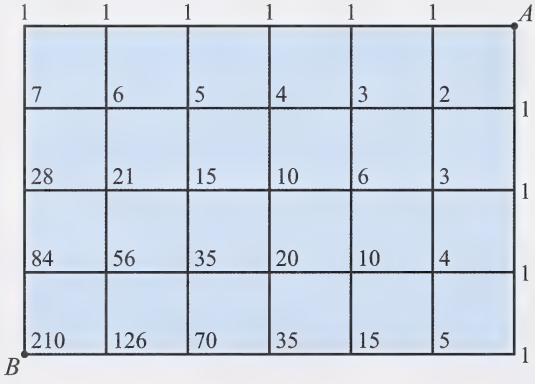
$$\therefore \text{ Number of paths} = \frac{12!}{7!5!}$$
$$= 792$$

There are 792 different paths from the northeast corner to the southwest corner.

5. a. Textbook questions 2 to 4 of "Practice," pp. 352 and 353

2. Method 1: Using Pascal's Triangle

Apply the techniques of Pascal's triangle to determine the number of routes to each point of intersection.



There are 210 pathways from A to B.

Method 2: Using Distinguishable Arrangements

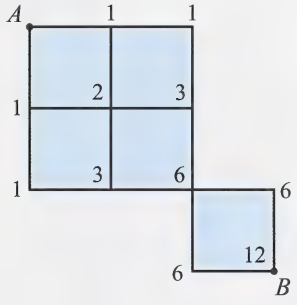
To travel from point A to point B, one must travel four blocks south and six blocks west. Therefore, the number of arrangements of S S S S W W W W W W is equal to the number of pathways.

$$\therefore \text{ Number of arrangements} = \frac{10!}{4!6!} \quad \leftarrow \text{There are 4 Ss and 6 Ws.}$$
$$= 210$$

There are 210 pathways from A to B.

3. Method 1: Using Pascal's Triangle

Determine the number of routes to each point of intersection by applying Pascal's triangle.



There are 12 pathways from A to B.

Method 2: Using Distinguishable Arrangements and the Fundamental Counting Principle

The number of pathways to the point where the two rectangles are joined is the number of distinguishable arrangements of E E S S.

$$\therefore \text{ Number of pathways} = \frac{4!}{2!2!}$$
$$= 6$$

There are 2 pathways from the point where the two rectangles are joined to B.

Therefore, using the Fundamental Counting Principle, there is a total of $6 \times 2 = 12$ pathways from A to B.

Section 2: Activity 1 (continued)

4. Method 1: Using Pascal's Triangle

The number of pathways from A to B is determined by showing the number of routes at each point of intersection.

There are 480 pathways from A to B.

Method 2: Using Distinguishable Arrangements and the Fundamental Counting Principle

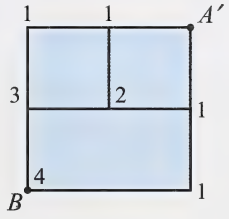
The number of pathways from A to the first point where the first rectangle and the second rectangle join is the number of distinguishable arrangements of S S S W W W.

$$\therefore \text{ Number of pathways} = \frac{6!}{3!3!}$$
$$= 20$$

The number of pathways from the beginning of the second rectangle to the point where the second rectangle and last rectangle join is the number of distinguishable arrangements of S S W W.

$$\therefore \text{ Number of pathways} = \frac{4!}{2!2!}$$
$$= 6$$

The number of pathways from the beginning of the last rectangle, A', to point B must be counted. The following diagram shows the number of routes.



Using the Fundamental Counting Principle,

$$\text{Number of pathways} = 20 \times 6 \times 4$$
$$= 480$$

There are 480 pathways from A to B.

Section 2: Activity 1 (continued)

b. Textbook questions 5 to 8 of "Applications and Problem Solving," p. 353

5. The first ten rows of Pascal's triangle are as follows.

Remember: Pascal's triangle starts with Row 0.

```
                                1
                              1   1
                            1   2   1
                          1   3   3   1
                        1   4   6   4   1
                      1   5   10  10  5   1
                    1   6   15  20  15  6   1
                  1   7   21  35  35  21  7   1
                1   8   28  56  70  56  28  8   1
              1   9   36  84  126 126 84  36  9   1
```

a.

Row, n	Sum of the Numbers in the Row	Row, n	Sum of the Numbers in the Row
0	1	5	32
1	1 + 1 = 2	6	64
2	4	7	128
3	8	8	256
4	16	9	512

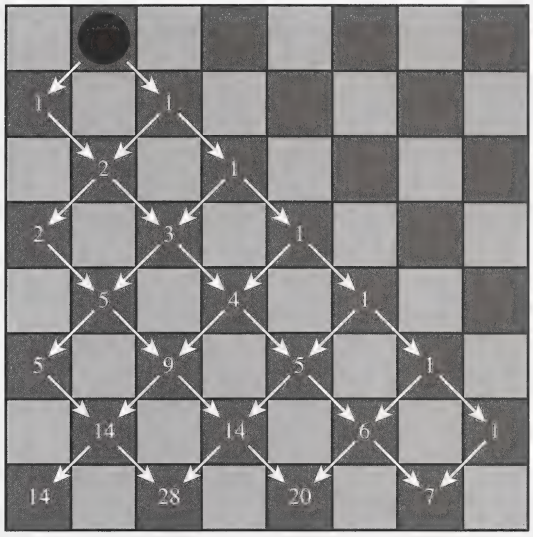
b. The prediction for the sum of the numbers in row 11, where $n = 10$, is 2^{10} or 1024.

Check the prediction.

$$\begin{aligned}\text{Sum of row } 11 &= 1+10+45+120+210+252+210+120+45+10+1\\ &= 1024\end{aligned}$$

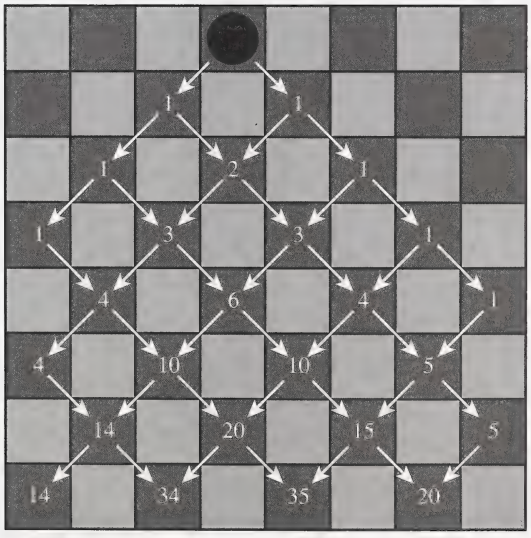
c. The sum of the numbers in the $(n + 1)$st row is 2^n.

6. a. The possible paths are indicated on the checkerboard shown.



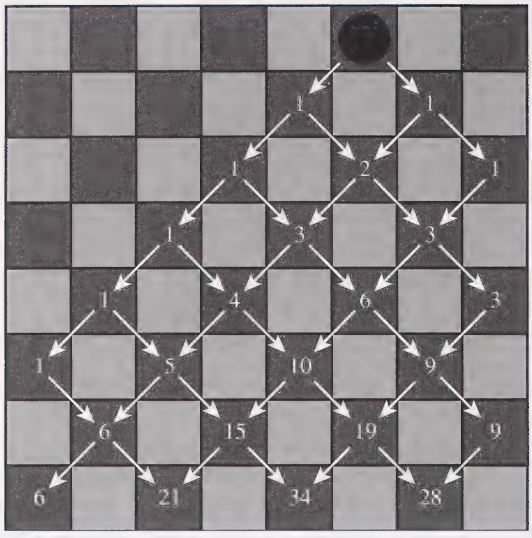
There are 14 + 28 + 20 + 7 = 69 possible paths for the checker to reach the opposite side of the gameboard.

b.

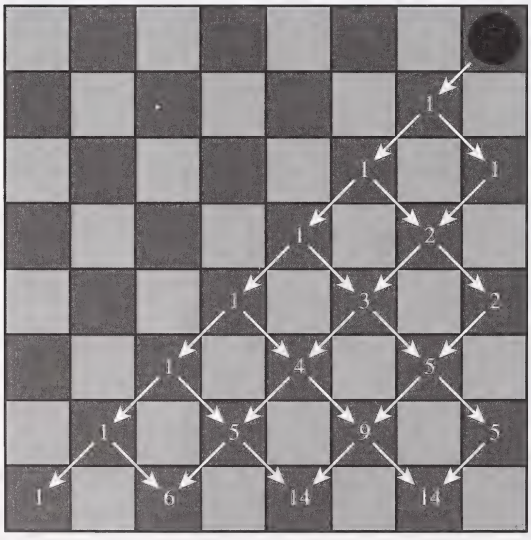


Here, there are 14 + 34 + 35 + 20 = 103 possible paths.

Section 2: Activity 1 (continued)



Here, there are $6 + 21 + 34 + 28 = 89$ possible paths.



Here, there are $1 + 6 + 14 + 14 = 35$ possible paths.

c. The most pathways occur when the checker starts in the second starting square from the left; therefore, from this starting point, the checker has the most chances to be "king."

7. a.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

$_4C_2$
$_4C_2$ $_4C_2$
$_4C_2$ $_4C_2$ $_4C_2$
$_4C_2$ $_4C_2$ $_4C_2$ $_4C_2$
$_4C_2$ $_4C_2$ $_4C_2$ $_4C_2$ $_4C_2$
$_4C_2$ $_4C_2$ $_4C_2$ $_4C_2$ $_4C_2$ $_4C_2$

b. The third entry in the fifth row is $_4C_2$.

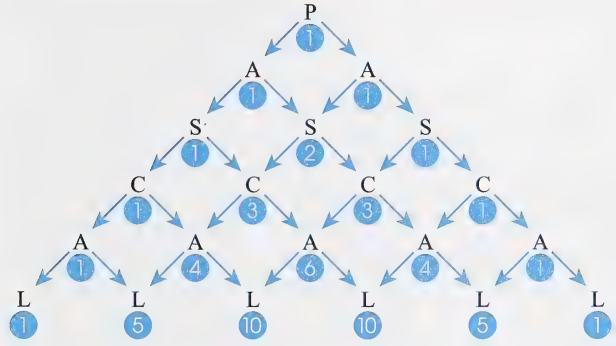
c. The relationship among $_4C_2$ and $_3C_1$ and $_4C_2$ is $_4C_2 = {}_3C_1 + {}_3C_2$.

d. $_nC_r + {}_nC_{r+1} = {}_{n+1}C_{r+1}$

This relationship is called Pascal's Theorem.

e. $_3C_2 + {}_3C_3 = {}_4C_3$
$_5C_3 + {}_5C_4 = {}_6C_4$
$_{10}C_6 + {}_{10}C_7 = {}_{11}C_7$

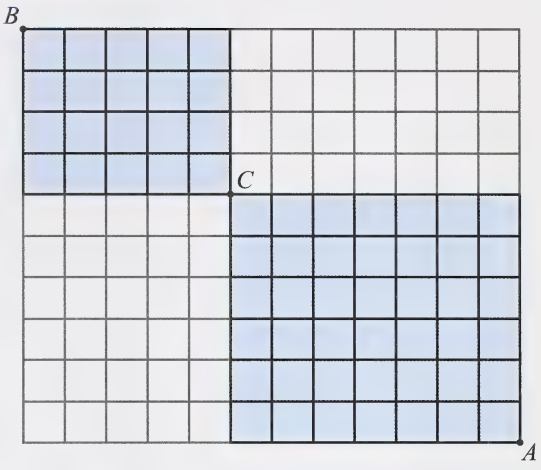
8. The triangular letter arrangement of the word PASCAL are as follows.



There are 32 paths that will spell PASCAL.

Section 2: Activity 1 (continued)

6. a. Grey out the areas of the diagram that cannot be used due to backtracking.



You can use Pascal's triangle to solve this problem, but it will take much longer and will increase the chance of error. So, in this case, it would be better to use distinguishable arrangements and the Fundamental Counting Principle.

Break the problem into two smaller parts.

First, to go from A to C, you need to go 7 blocks left and 6 blocks up. this corresponds to the number of arrangements of L L L L L L L U U U U U U.

$$\therefore \text{ Number of paths from } A \text{ to } C = \frac{13!}{7!6!}$$
$$= 1716$$

Next, to go from C to B, you need to go 5 blocks left and 4 blocks up. This corresponds to the number of arrangements of L L L L L U U U U.

$$\therefore \text{ Number of paths from } C \text{ to } B = \frac{9!}{5!4!}$$
$$= 126$$

Using the Fundamental Counting Principle,

$$\text{Number of paths from } A \text{ to } B = 1716 \times 126$$
$$= 216\,216$$

There are 216 216 paths from A to B that pass through C.

b. The total number of paths from A to B requires you to go 12 blocks left and 10 blocks up. This corresponds to the number of arrangements of L L L L L L L L L L L U U U U U U U U U U U.

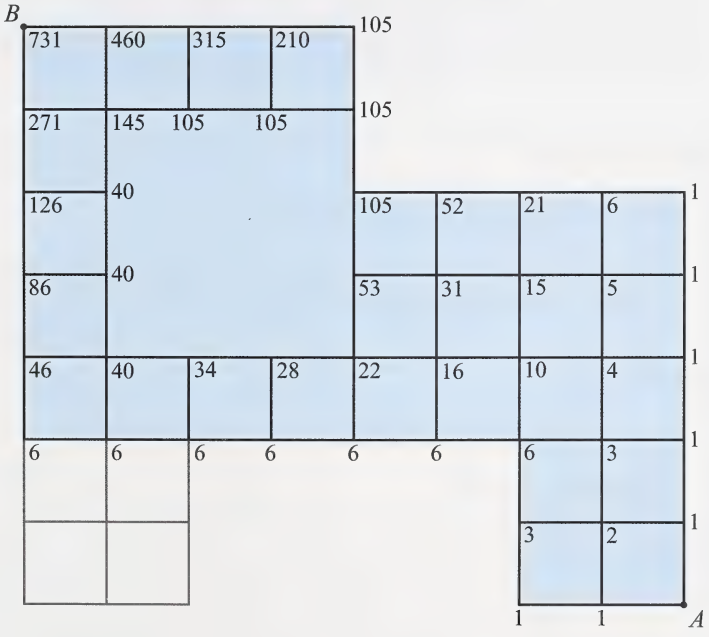
$$\therefore \text{ Total Number of paths from } A \text{ to } B = \frac{22!}{12!10!}$$
$$= 646\,646$$

Calculate the percentage of the total number of paths that pass through C.

$$\text{Percentage} = \frac{\text{number of paths through } C}{\text{total number of paths}} \times 100\%$$
$$= \frac{216\,216}{646\,646} \times 100\%$$
$$= 33.4\%$$

Approximately 33.4% of the total number of paths pass through C.

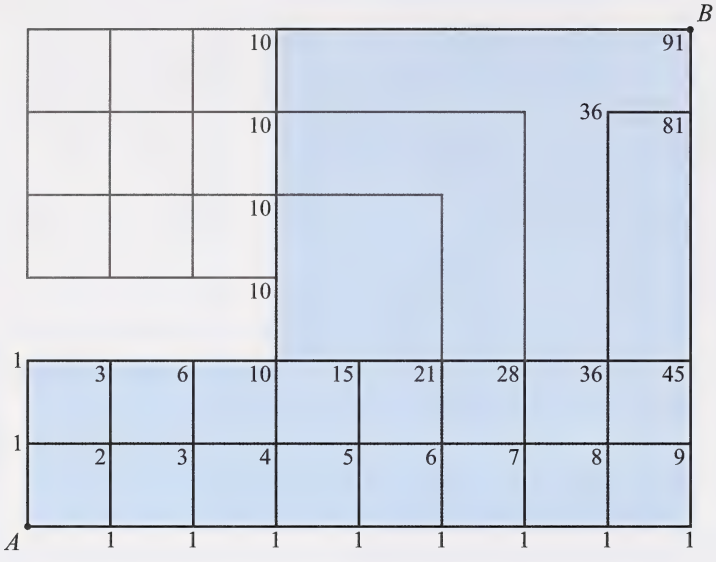
7. a. Use the techniques of Pascal's triangle to solve this problem. the greyed out areas of the diagram show the paths that cannot be used because they involve backtracking.



There are 731 paths from A to B.

Section 2: Activity 1 (continued)

b. The greyed out areas of the diagram show the paths that cannot be used because they involve backtracking.



There are 91 paths from A to B.

8. a. Use distinguishable arrangements to solve this problem.

To go from C to D, you must travel 2 blocks toward the front, 5 blocks left, and 5 blocks down. This is equivalent to the number of distinguishable arrangements of F F L L L L L D D D D D.

$$\therefore \text{ Number of paths} = \frac{12!}{2!5!5!}$$
$$= 16\,632$$

There are 16 632 paths from C to D.

b. Use distinguishable arrangements and the Fundamental Counting Principle to solve this problem.

Break the problem into two smaller parts.

First, to go from C to E, you must travel 2 blocks up, 3 blocks left, and 3 blocks toward the back. This is equivalent to the number of distinguishable arrangements of U U L L L B B B.

$$\therefore \text{ Number of paths from } C \text{ to } E = \frac{8!}{2!3!3!}$$
$$= 560$$

Next, to go from E to D, you must travel 3 blocks up, 2 blocks left, and 3 blocks toward the back. This is equivalent to the number of distinguishable arrangements of U U U L L B B B.

$$\therefore \text{ Number of paths from } E \text{ to } D = \frac{8!}{3!2!3!}$$
$$= 560$$

Using the Fundamental Counting Principle,

$$\text{Number of paths from } C \text{ to } D = 560 \times 560$$
$$= 313\,600$$

There are 313 600 paths from C to D.

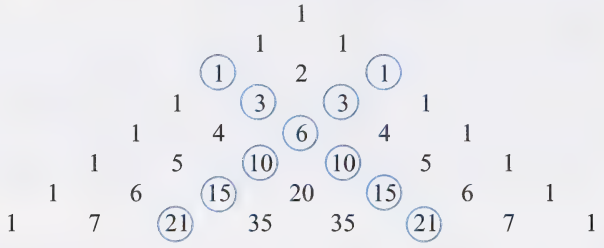
9. The number of paths from A to B in both diagrams is the same because there is still only one link between the two cubes.

Section 2: Activity 1 (continued)

10. a. The fourth and fifth triangular numbers are as follows.

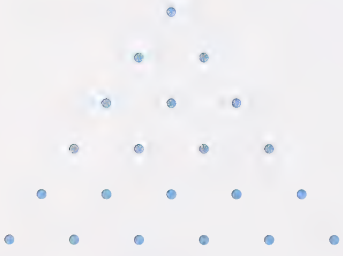
b. The value of the fourth triangular number is 10, and the value of the fifth triangular number is 15.

c. The first eight rows of Pascal's triangle are as follows. The triangular numbers have been circled. Because of symmetry, there are two diagonals along which the triangular numbers lie.



d. The sixth triangular number will be 21.

Check



The value of the sixth triangular number is 21.

e.

$${}_0C_0$$

$${}_1C_0 \quad {}_1C_1$$

$${}_2C_0 \quad {}_2C_1 \quad {}_2C_2$$

$${}_3C_0 \quad {}_3C_1 \quad {}_3C_2 \quad {}_3C_3$$

$${}_4C_0 \quad {}_4C_1 \quad {}_4C_2 \quad {}_4C_3 \quad {}_4C_4$$

$${}_5C_0 \quad {}_5C_1 \quad {}_5C_2 \quad {}_5C_3 \quad {}_5C_4 \quad {}_5C_5$$

$${}_6C_0 \quad {}_6C_1 \quad {}_6C_2 \quad {}_6C_3 \quad {}_6C_4 \quad {}_6C_5 \quad {}_6C_6$$

$${}_7C_0 \quad {}_7C_1 \quad {}_7C_2 \quad {}_7C_3 \quad {}_7C_4 \quad {}_7C_5 \quad {}_7C_6 \quad {}_7C_7$$

The first eight triangular numbers are ${}_2C_2$, ${}_3C_2$, ${}_4C_2$, ${}_5C_2$, ${}_6C_2$, ${}_7C_2$, ${}_8C_2$, and ${}_9C_2$, or ${}_2C_0$, ${}_3C_1$, ${}_4C_2$, ${}_5C_3$, ${}_6C_4$, ${}_7C_5$, ${}_8C_6$, and ${}_9C_7$.

The general expression for the nth triangular number is ${}_{n+1}C_2$ or ${}_{n+1}C_{n-1}$.

Section 2: Activity 2

1. a. Textbook question "Explore: Look for a Pattern," p. 354

$$(a+b)^0 = 1$$

$$(a+b)^1 = a+b$$

$$\begin{aligned}(a+b)^2 &= (a+b)(a+b)\\ &= a^2+ab+ab+b^2\\ &= a^2+2ab+b^2\end{aligned}$$

$$\begin{aligned}(a+b)^3 &= (a+b)(a+b)(a+b)\\ &= (a^2+2ab+b^2)(a+b)\\ &= a^3+a^2b+2a^2b+2ab^2+ab^2+b^3\\ &= a^3+3a^2b+3ab^2+b^3\end{aligned}$$

$$\begin{aligned}(a+b)^4 &= (a+b)(a+b)(a+b)(a+b)\\ &= (a^3+3a^2b+3ab^2+b^3)(a+b)\\ &= a^4+a^3b+3a^3b+3a^2b^2+3a^2b^2+3ab^3+ab^3+b^4\\ &= a^4+4a^3b+6a^2b^2+4ab^3+b^4\end{aligned}$$

Section 2: Activity 2 (continued)

b. Textbook questions 1 to 5 of "Inquire," p. 354

1. The coefficients are as follows:

$(a+b)^0$					1				
$(a+b)^1$				1		1			
$(a+b)^2$			1		2		1		
$(a+b)^3$		1		3		3		1	
$(a+b)^4$	1		4		6		4		1

The pattern of the numerical coefficients of the expansions are the same numbers as the numbers in Pascal's triangle.

2. The numerical coefficients of successive terms in the expansion of $(a+b)^5$ are 1, 5, 10, 10, 5, and 1.

3. Assume the terms in the binomial expansion are written with descending powers of a.

The powers of a begin with the same exponent as the power of the binomial and decrease by 1 until a^0 is reached.

The powers of b begin with b^0 and increase by 1 until b has the same exponent as the exponent of the power of the binomial.

The power of the binomial (or degree) is the same as the numerical coefficient of the second term in the expansion.

A prediction for the expansion of $(a+b)^5$ is as follows:

$$1a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

4.
$$\begin{aligned}(a+b)^5 &= (a+b)^4(a+b)\\ &= (a^4+4a^3b+6a^2b^2+4ab^3+b^4)(a+b)\\ &= a^5+a^4b+4a^4b+4a^3b^2+6a^3b^2+6a^2b^3+4a^2b^3+4ab^4+ab^4+b^5\\ &= a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5\end{aligned}$$

The prediction is correct.

5. A general term of $(a+b)^n$ is ${}_nC_r a^{n-r}b^r$.

2. Textbook questions 1, 2, 5, and 10 of "Practice," p. 356

1. The row of Pascal's triangle where the second entry matches the exponent 5 is the correct row for obtaining the numerical coefficients of the expansion.

$$\begin{array}{ccccccccccc} & & & & & 1 & & & & & \\ & & & & 1 & & 1 & & & & \\ & & & 1 & & 2 & & 1 & & & \\ & & 1 & & 3 & & 3 & & 1 & & \\ & 1 & & 4 & & 6 & & 4 & & 1 & \\ 1 & & 5 & & 10 & & 10 & & 5 & & 1 \end{array}$$

The expansion is of the form

$$(a+b)^5 = 1a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + 1b^5$$

Because $(a+b)^5 = (x+1)^5$, substitute x for a and 1 for b into the preceding expansion, and simplify.

$$\begin{aligned}(x+1)^5 &= 1x^5 + 5x^4(1) + 10x^3(1)^2 + 10x^2(1)^3 + 5x(1)^4 + 1(1)^5 \\ &= x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1\end{aligned}$$

2. The row of Pascal's triangle where the second entry matches the exponent 4 is the correct row.

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

The expansion is of the form

$$(a+b)^4 = 1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4$$

Substitute x for a and -2 for b into the expansion, and simplify.

$$\begin{aligned}(x-2)^4 &= 1x^4 + 4x^3(-2) + 6x^2(-2)^2 + 4x(-2)^3 + 1(-2)^4 \\ &= x^4 - 8x^3 + 6x^2(4) + 4x(-8) + 16 \\ &= x^4 - 8x^3 + 24x^2 - 32x + 16\end{aligned}$$

Section 2: Activity 2 (continued)

5. The row of Pascal's triangle where the second entry matches the exponent 4 is the correct row.

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

$\therefore (a+b)^4 = 1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4$

Substitute $4x$ for a and 1 for b into the expansion, and simplify.

$$\begin{aligned}(4x+1)^4 &= 1(4x)^4 + 4(4x)^3(1) + 6(4x)^2(1)^2 + 4(4x)(1)^3 + 1(1)^4 \\ &= 256x^4 + 4(64x^3) + 6(16x^2) + 16x + 1 \\ &= 256x^4 + 256x^3 + 96x^2 + 16x + 1\end{aligned}$$

10. The row of Pascal's triangle where the second entry matches the exponent 3 is the correct row.

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 1 & & 2 & & 1 & \\ 1 & & 3 & & 3 & & 1 \end{array}$$

$\therefore (a+b)^3 = 1a^3 + 3a^2b + 3ab^2 + 1b^3$

Substitute $4a$ for a and $-3b$ for b into the expansion, and simplify.

$$\begin{aligned}(4a-3b)^3 &= 1(4a)^3 + 3(4a)^2(-3b) + 3(4a)(-3b)^2 + 1(-3b)^3 \\ &= 64a^3 + 3(16a^2)(-3b) + 3(4a)(9b^2) - 27b^3 \\ &= 64a^3 - 144a^2b + 108ab^2 - 27b^3\end{aligned}$$

3. a. Textbook questions 12, 13, 15, 16, 27, 28, 29, and 30 of "Practice," pp. 356 and 357

12. Use the Binomial Theorem, where $a = 2x$, $b = 1$, and $n = 3$.

$$\begin{aligned}(2x+1)^3 &= {}_3C_0(2x)^3(1)^0 + {}_3C_1(2x)^2(1)^1 + {}_3C_2(2x)^1(1)^2 + {}_3C_3(2x)^0(1)^3 \\ &= 1(8x^3)(1) + 3(4x^2)(1) + 3(2x)(1) + 1(1)(1) \\ &= 8x^3 + 12x^2 + 6x + 1\end{aligned}$$

13. $(x-3)^5 = [x+(-3)]^5$

Use the Binomial Theorem, where $a = x$, $b = -3$, and $n = 5$.

$$\begin{aligned}(x-3)^5 &= {}_5C_0\,(x)^5(-3)^0 + {}_5C_1\,(x)^4(-3)^1 + {}_5C_2\,(x)^3(-3)^2 + {}_5C_3\,(x)^2(-3)^3 \\ &\quad + {}_5C_4\,(x)^1(-3)^4 + {}_5C_5\,(x)^0(-3)^5 \\ &= 1(x^5)(1) + 5(x^4)(-3) + 10(x^3)(9) + 10(x^2)(-27) + 5(x)(81) + (1)(1)(-243) \\ &= x^5 - 15x^4 + 90x^3 - 270x^2 + 405x - 243\end{aligned}$$

15. Use the Binomial Theorem, where $a = 2$, $b = x$, and $n = 3$.

$$\begin{aligned}(2+x)^3 &= {}_3C_0\,(2)^3(x)^0 + {}_3C_1\,(2)^2(x)^1 + {}_3C_2\,(2)^1(x)^2 + {}_3C_3\,(2)^0(x)^3 \\ &= 1(8)(1) + 3(4)(x) + 3(2)(x^2) + 1(1)(x^3) \\ &= 8 + 12x + 6x^2 + x^3\end{aligned}$$

16. Use the Binomial Theorem, where $a = 4x$, $b = y$, and $n = 5$.

$$\begin{aligned}(4x+y)^5 &= {}_5C_0\,(4x)^5(y)^0 + {}_5C_1\,(4x)^4(y)^1 + {}_5C_2\,(4x)^3(y)^2 + {}_5C_3\,(4x)^2(y)^3 \\ &\quad + {}_5C_4\,(4x)^1(y)^4 + {}_5C_5\,(4x)^0(y)^5 \\ &= 1(1024x^5)(1) + 5(256x^4)(y) + 10(64x^3)(y^2) + 10(16x^2)(y^3) \\ &\quad + 5(4x)(y^4) + (1)(1)(y^5) \\ &= 1024x^5 + 1280x^4y + 640x^3y^2 + 160x^2y^3 + 20xy^4 + y^5\end{aligned}$$

27. $a = x$, $b = 2y$, $n = 6$, and $r + 1 = 4$

$$r = 3$$

$$\begin{aligned}\therefore\ t_{r+1} &= {}_nC_r\,a^{n-r}b^r \\ t_{3+1} &= {}_6C_3\,(x)^{6-3}(2y)^3 \\ t_4 &= {}_6C_3\,(x)^3(2y)^3 \\ &= 20(x^3)(8y^3) \\ &= 160x^3y^3\end{aligned}$$

The fourth term of $(x + 2y)^6$ is $160x^3y^3$.

Section 2: Activity 2 (continued)

28. $a = 2x,\ b = 3y,\ n = 7,$ and $r + 1 = 3$

$r = 2$

$$\therefore\ t_{r+1} = {}_nC_r a^{n-r} b^r$$
$$t_{2+1} = {}_7C_2 (2x)^{7-2} (3y)^2$$
$$t_3 = {}_7C_2 (2x)^5 (3y)^2$$
$$= 21(32x^5)(9y^2)$$
$$= 6048x^5y^2$$

The third term of $(2x + 3y)^7$ is $6048x^5y^2$.

29. $(3x - 7y)^5 = \left[3x + (-7y)\right]^5$

$\therefore a = 3x,\ b = -7y,\ n = 5,$ and $r + 1 = 4$

$r = 3$

$$\therefore\ t_{r+1} = {}_nC_r a^{n-r} b^r$$
$$t_{3+1} = {}_5C_3 (3x)^{5-3} (-7y)^3$$
$$t_4 = {}_5C_3 (3x)^2 (-7y)^3$$
$$= 10(9x^2)(-343y^3)$$
$$= -30\ 870x^2y^3$$

The fourth term of $(3x - 7y)^5$ is $-30\ 870x^2y^3$.

30. $a = 2x,\ b = y,\ n = 13,$ and $r + 1 = 11$

$r = 10$

$$\therefore\ t_{r+1} = {}_nC_r a^{n-r} b^r$$
$$t_{10+1} = {}_{13}C_{10} (2x)^{13-10} (y)^{10}$$
$$t_{11} = {}_{13}C_{10} (2x)^3 (y)^{10}$$
$$= 286(8x^3)(y^{10})$$
$$= 2288x^3y^{10}$$

The eleventh term of $(2x + y)^{13}$ is $2288x^3y^{10}$.

b. Textbook questions 31, 33, 36, 37, and 39 of "Applications and Problem Solving," p. 357

31. The general term in the expansion of $(3x + 1)^8$ is

$$\begin{aligned} t_{r+1} &= {}_nC_r a^{n-r} b^r \\ &= {}_8C_r (3x)^{8-r} (1)^r \\ &= {}_8C_r (3x)^{8-r} \end{aligned}$$

Use guess and check to find the largest numerical coefficient.

Try $r = 0$.

$$\begin{aligned} t_{0+1} &= {}_8C_0 (3x)^{8-0} \\ t_1 &= {}_8C_0 (3x)^8 \\ &= 1(6561x^8) \\ &= 6561x^8 \end{aligned}$$

Try $r = 1$.

$$\begin{aligned} t_{1+1} &= {}_8C_1 (3x)^{8-1} \\ t_2 &= {}_8C_1 (3x)^7 \\ &= 8(2187x^7) \\ &= 17\,496x^7 \end{aligned}$$

Try $r = 2$.

$$\begin{aligned} t_{2+1} &= {}_8C_2 (3x)^{8-2} \\ t_3 &= {}_8C_2 (3x)^6 \\ &= 28(729x^6) \\ &= 20\,412x^6 \end{aligned}$$

Try $r = 3$.

$$\begin{aligned} t_{3+1} &= {}_8C_3 (3x)^{8-3} \\ t_4 &= {}_8C_3 (3x)^5 \\ &= 56(243x^5) \\ &= 13\,608x^5 \end{aligned}$$

After $r = 2$, the numerical coefficients decrease. Therefore, the largest numerical coefficient in the expansion of $(3x + 1)^8$ is 20 412.

Section 2: Activity 2 (continued)

33. a. $k^3+3k^2m+3km^2+m^3$

$=1(k^3)(m^0)+3(k^2)(m^1)+3(k^1)(m^2)+1(k^0)(m^3)$

$={}_3C_0\,(k)^3\,(m)^0+{}_3C_1\,(k)^2\,(m)^1+{}_3C_2\,(k)^1\,(m)^2$
$\quad+{}_3C_3\,(k)^0\,(m)^3$

$=(k+m)^3$

Using the Binomial Theorem, $a=k$, $b=m$, and $n=3$.

$\therefore (a+b)^n=(k+m)^3$

b. $x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5$

$=1(x^5)(y^0)+5(x^4)(y^1)+10(x^3)(y^2)$
$\quad+10(x^2)(y^3)+5(x^1)(y^4)+1(x^0)(y^5)$

$={}_5C_0\,(x)^5\,(y)^0+{}_5C_1\,(x)^4\,(y)^1+{}_5C_2\,(x)^3\,(y)^2$
$\quad+{}_5C_3\,(x)^2\,(y)^3+{}_5C_4\,(x)^1\,(y)^4+{}_5C_5\,(x)^0\,(y)^5$

$=(x+y)^5$

Using the Binomial Theorem, $a=x$, $b=y$, and $n=5$.

$\therefore (a+b)^n=(x+y)^5$

c. $x^3+6x^2+12x+8={}_3C_0\,(x)^3\,(b)^0+{}_3C_1\,(x)^2\,(b)^1+{}_3C_2\,(x)^1\,(b)^2+{}_3C_3\,(x)^0\,(b)^3$

$=(x+b)^3$

Use any term, other than the first term, and the general term to find the value of b. For example, use the second term.

$a=x,\ n=3,$ and $r+1=2$

$r=1$

$\therefore\ t_{r+1}={}_nC_r\,a^{n-r}b^r$

$t_{1+1}={}_3C_1\,(x)^{3-1}\,b^1$

$t_2={}_3C_1\,(x)^2\,b^1$

$6x^2=3(x^2)b$

$6x^2=3x^2b$

$b=2$

$\therefore x^3+6x^2+12x+8=(x+2)^3$

36. To determine whether the expansion of $\left(x^2-\frac{1}{x}\right)^2$ contains an x^3 term, expand the binomial using the Binomial Theorem.

$a=x^2$, $b=\left(-\frac{1}{x}\right)$, and $n=2$

$$\therefore \left(x^2-\frac{1}{x}\right)^2 = {}_2C_0\left(x^2\right)^{2-0}\left(-\frac{1}{x}\right)^0 + {}_2C_1\left(x^2\right)^{2-1}\left(-\frac{1}{x}\right)^1 + {}_2C_2\left(x^2\right)^{2-2}\left(-\frac{1}{x}\right)^2$$

$$= 1\left(x^2\right)^2(1) + 2\left(x^2\right)^1\left(-\frac{1}{x}\right) + 1\left(x^2\right)^0\left(\frac{1}{x^2}\right)$$

$$= x^4 - \frac{2}{x} + \frac{1}{x^2}$$

There is no x^3 term in the expansion.

37. **a.** The first two terms of $(1+x)^n$ can be obtained by using the Binomial Theorem, where $a=1$ and $b=x$.

$$\therefore 1+5x = {}_nC_0(1)^{n-0}(x)^0 + {}_nC_1(1)^{n-1}(x)^1$$

$$1+5x = 1(1)(1) + \frac{n!}{(n-1)!1!}(1)(x)$$

$$1+5x = 1+\frac{n\cancel{(n-1)!}}{\cancel{(n-1)!}}(x)$$

$$1+5x = 1+nx$$

$$5x = nx$$

$$5 = n$$

$$n = 5$$

Section 2: Activity 2 (continued)

b. The first three terms in the expansion of $(1+a)^n$ are $t_1 = 1$, $t_2 = -18$, and $t_3 = 144$.

Use the general term to determine a and n.

$a = 1,\ b = a,\ n = n,$ and $r+1 = 2$

$r = 1$

$$\therefore\ t_{r+1} = {}_nC_r a^{n-r} b^r$$
$$t_{1+1} = {}_nC_1 (1)^{n-1} (a)^1$$
$$t_2 = \frac{n!}{(n-1)!1!}(1)(a)$$
$$-18 = \frac{n(n-1)!}{(n-1)!}(a)$$
$$-18 = na \quad (1)$$

$a = 1,\ b = a,\ n = n,$ and $r+1 = 3$

$r = 2$

$$\therefore\ t_{r+1} = {}_nC_r a^{n-r} b^r$$
$$t_{2+1} = {}_nC_2 (1)^{n-2} (a)^2$$
$$t_3 = \frac{n!}{(n-2)!2!}(1)(a^2)$$
$$144 = \frac{n(n-1)(n-2)!}{(n-2)!2!}(a^2)$$
$$144 = \frac{n(n-1)}{2}(a^2)$$
$$288 = n(n-1)(a^2) \quad (2)$$

Solve (1) for a.

$$-18 = na$$
$$a = -\frac{18}{n}$$

Substitute $-\frac{18}{n}$ for a into (2).

$$288 = n(n-1)(a^2)$$
$$288 = n(n-1)\left(-\frac{18}{n}\right)^2$$
$$288 = (n^2 - n)\left(\frac{324}{n^2}\right)$$
$$288 = 324 - \frac{324}{n}$$
$$\frac{324}{n} = 36$$
$$n = \frac{324}{36}$$
$$= 9$$

Substitute $n = 9$ into (1).

$$-18 = na$$
$$-18 = 9a$$
$$a = -2$$

Therefore, $a = -2$ and $n = 9$.

c. The first three terms in the expansion of $(1 - ax)^n$ are $t_1 = 1$, $t_2 = -12x$, and $t_3 = 63x^2$.

Use the general term to determine a and n.

$a = 1, b = -ax, n = n,$ and $r + 1 = 2$

$r = 1$

$$\therefore t_{r+1} = {}_nC_r a^{n-r} b^r$$
$$t_{1+1} = {}_nC_1 (1)^{n-1} (-ax)^1$$
$$t_2 = \frac{n!}{(n-1)!1!}(1)(-ax)$$
$$-12x = \frac{n(n-1)!}{(n-1)!}(-ax)$$
$$-12 = -na$$
$$12 = na \quad (1)$$

$a = 1, b = -ax, n = n,$ and $r + 1 = 3$

$r = 2$

$$\therefore t_{r+1} = {}_nC_r a^{n-r} b^r$$
$$t_{2+1} = {}_nC_2 (1)^{n-2} (-ax)^2$$
$$t_3 = \frac{n!}{(n-2)!2!}(1)(a^2x^2)$$
$$63x^2 = \frac{n(n-1)(n-2)!}{(n-2)!2!}(a^2x^2)$$
$$63 = \frac{n(n-1)}{2}(a^2)$$
$$126 = n(n-1)(a^2) \quad (2)$$

Solve (1) for a.

$$12 = na$$
$$a = \frac{12}{n}$$

Section 2: Activity 2 (continued)

Substitute $\frac{12}{n}$ for a into (2).

$$126 = n(n-1)(a^2)$$
$$126 = n(n-1)\left(\frac{12}{n}\right)^2$$
$$126 = (n^2 - n)\left(\frac{144}{n^2}\right)$$
$$126 = 144 - \frac{144}{n}$$
$$\frac{144}{n} = 18$$
$$n = \frac{144}{18}$$
$$= 8$$

Substitute $n = 8$ into (1).

$$12 = na$$
$$12 = 8a$$
$$a = \frac{12}{8}$$
$$= \frac{3}{2} \text{ or } 1.5$$

Therefore, $a = \frac{3}{2}$ or 1.5 and $n = 8$.

39. Use the general term of $\left(x+\frac{1}{2x}\right)^9$, where $a=x$, $b=\frac{1}{2x}$, and $n=9$.

$$t_{r+1} = {}_nC_r a^{n-r} b^r$$

$$t_{r+1} = {}_9C_r (x)^{9-r} \left(\frac{1}{2x}\right)^r$$

$$= {}_9C_r x^{9-r} \frac{1}{2^r x^r}$$

$$= \frac{{}_9C_r}{2^r} \times \frac{x^{9-r}}{x^r}$$

$$\therefore \frac{x^{9-r}}{x^r} = x^7$$

$$x^{9-r-r} = x^7$$

$$9-2r = 7$$

$$2 = 2r$$

$$1 = r$$

You can ignore the numerical coefficient $\frac{{}_9C_r}{2^r}$.

The term involving x^7 occurs when $r = 1$.

$$\therefore t_{1+1} = \frac{{}_9C_1}{2^1} \times \frac{x^{9-1}}{x^1}$$

$$t_2 = \frac{9}{2}x^7$$

The second term is $\frac{9}{2}x^7$.

4. Textbook questions 2.a. and 2.b. of Investigation 2, "Classification by Mathematical Operations," p. 359

2. a. *keyword letters*: M ATHM A THMATH
message letters: I HAVE A SECRET
enciphered letters: U HTC**Q** **A L L O R X A**

Section 2: Activity 2 (continued)

b. *keyword letters*: KEYWO RDK EYWORDKE YWOR DKEYWOR DKEYWOR DKE YWORDKE

enciphered letters: MSSHR PRE HCYWGKOV RDWJ POWQWUV ZSXFKIK WRI IAMNRBH

message letters: COULD YOU DECIPHER THIS MESSAGE WITHOUT THE KEYWORD

The message is "Could you decipher this message without the keyword?"

Section 2: Follow-up Activities

Extra Help

1. Textbook questions 1, 2, 3, 4, 6, 8, 13, and 14 of "Mental Math," p. 333

1. $(2x)^3 = 2^3 x^3$
$= 8x^3$

2. $(27x^6)^{\frac{1}{3}} = 27^{\frac{1}{3}}(x^6)^{\frac{1}{3}}$
$= 3x^2$

$27^{\frac{1}{3}} = \sqrt[3]{27}$

3. $(3a)^2(4a^2)^3 = (3^2a^2)\left[4^3(a^2)^3\right]$
$= (9a^2)(64a^6)$
$= 576a^{2+6}$
$= 576a^8$

4. $\left(\frac{2}{x}\right)^3 = \frac{2^3}{x^3}$
$= \frac{8}{x^3}$

6. $(xy)^3(x^2y)^3 = x^3y^3(x^2)^3y^3$
$= x^3y^3x^6y^3$
$= x^{3+6}y^{3+3}$
$= x^9y^6$

8. $\left(\frac{3}{b^2}\right)^{-2} = \frac{3^{-2}}{(b^2)^{-2}}$
$= \frac{3^{-2}}{b^{-4}}$
$= \frac{\frac{1}{9}}{\frac{1}{b^4}}$
$= \frac{b^4}{9}$

13. $\dfrac{x^3}{y^4} \times \dfrac{y^6}{x^5} = \dfrac{x^3 y^6}{x^5 y^4}$

$= x^{3-5} y^{6-4}$

$= x^{-2} y^2$

$= \dfrac{y^2}{x^2}$

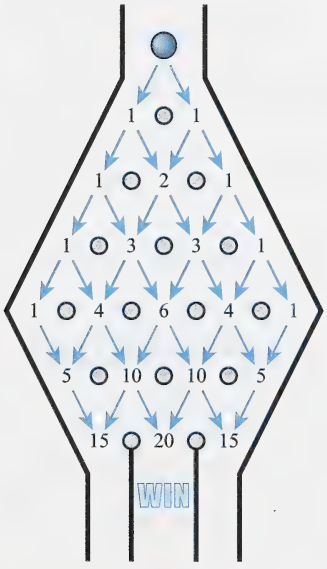
14. $\dfrac{y+y^2}{x^2} \times \dfrac{x}{y} = \dfrac{y(1+y)}{x^2} \times \dfrac{x}{y}$

$= \dfrac{y(1+y)x}{x^2 y}$

$= \dfrac{(1+y)}{x}$

2. Textbook questions 9, 10, 11, 12.a., 12.d., 13.b., and 13.d. of "Review," pp. 360 and 361

9. a.

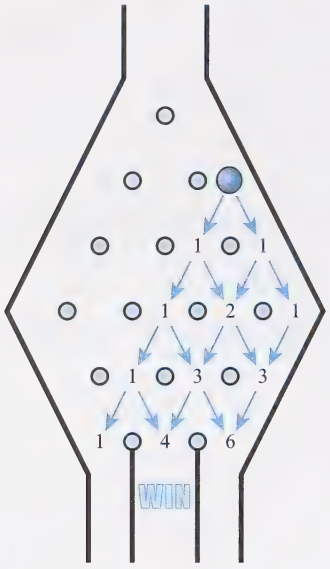


There are 20 paths that lead to a win.

b. There are 15 + 15 = 30 paths that lead to a loss.

Extra Help (continued)

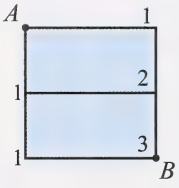
c.



There are 4 pathways that would lead to a win if you started at position X.

10. a. Method 1: Using Pascal's Triangle

Using the diagram, record the number of paths to each vertex.



There are 3 paths from A to B.

Method 2: Using Distinguishable Arrangements

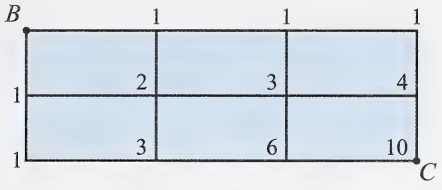
The path from A to B consists of three blocks: two south and one east. The number of pathways is equal to the number of distinguishable arrangements of S S E.

$$\therefore \text{ Number of pathways} = \frac{3!}{2!}$$
$$= 3$$

There are three paths from A to B.

b. Method 1: Using Pascal's Triangle

Record the number of paths to each vertex.



There are 10 paths from B to C.

Method 2: Using Distinguishable Arrangements

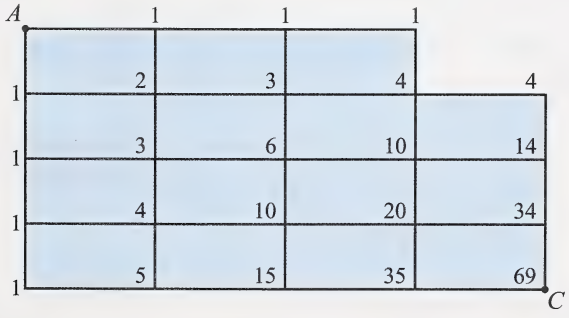
To travel from B to C, you must travel a total of five blocks: three blocks east and two blocks south.

$$\therefore \text{ Number of pathways} = \frac{5!}{3!2!}$$
$$= 10$$

There are 10 paths from B to C.

c. The upper-right rectangle in the following diagram is missing because that pathway is blocked by the river.

Record the number of paths to each vertex.



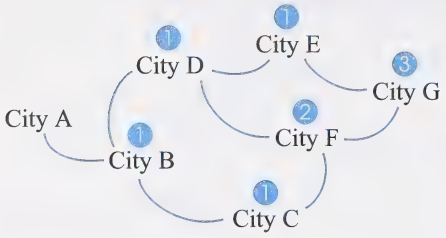
There are 69 paths from A to C.

Extra Help (continued)

d. The sum of the paths from A to B and then B to C is $3 + 10 = 13$.

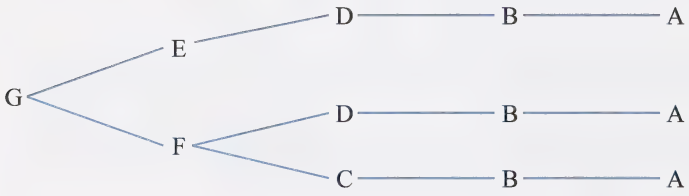
The sum would not represent the paths from A to C. If you wanted the paths from A to B and then B to C, you would use the Fundamental Counting Principle. Even so, the number of paths beginning at A and ending at C and going through B is $3 \times 10 = 30$. There are paths from A to C that do not go through B.

11. a.



There are 3 routes from City A to City G.

b.



There are 3 routes from City G to City A. Therefore, there are $3 \times 3 = 9$ routes from City A to City G and back to City A.

c. Using the diagram in the answer to question 11.b., there are two paths from City G to City D to City A.

12. a.
$$\begin{aligned}(5+x)^3 &= {}_3C_0\,(5)^3\,(x)^0 + {}_3C_1\,(5)^2\,(x)^1 + {}_3C_2\,(5)^1\,(x)^2 + {}_3C_3\,(5)^0\,(x)^3 \\ &= (1)(125)(1) + (3)(25)(x) + (3)(5)(x^2) + (1)(1)(x^3) \\ &= 125 + 75x + 15x^2 + x^3\end{aligned}$$

d.
$$\begin{aligned}(x-3y)^4 &= \left[x + (-3y)\right]^4 \\ &= {}_4C_0\,(x)^4\,(-3y)^0 + {}_4C_1\,(x)^3\,(-3y)^1 + {}_4C_2\,(x)^2\,(-3y)^2 + {}_4C_3\,(x)^1\,(-3y)^3 \\ &\quad + {}_4C_4\,(x)^0\,(-3y)^4 \\ &= 1(x^4)(1) + 4(x^3)(-3y) + 6(x^2)(9y^2) + 4(x)(-27y^3) + 1(1)(81y^4) \\ &= x^4 - 12x^3y + 54x^2y^2 - 108xy^3 + 81y^4\end{aligned}$$

13. b. $a=3, b=x, n=4,$ and $r+1=2$

$r=1$

$$\therefore\ t_{r+1} = {}_nC_r a^{n-r} b^r$$
$$t_{1+1} = {}_4C_1 (3)^{4-1} (x)^1$$
$$t_2 = 4(3^3)(x)$$
$$= 4(27)x$$
$$= 108x$$

d. $a=x, b=y, n=7,$ and $r+1=4$

$r=3$

$$\therefore\ t_{r+1} = {}_nC_r a^{n-r} b^r$$
$$t_{3+1} = {}_7C_3 (x)^{7-3} (y)^3$$
$$t_4 = 35(x)^4 (y)^3$$
$$= 35x^4 y^3$$

Enrichment

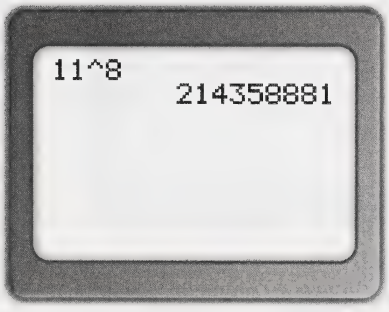
1. a. You can determine the value of 11^8 using the values in row 8 of Pascal's triangle.

Row 8 of Pascal's triangle is as follows:

1 8 28 56 70 56 28 8 1

$$\therefore\ 11^8 = 1(10)^8 + 8(10)^7 + 28(10)^6 + 56(10)^5 + 70(10)^4 + 56(10)^3 + 28(10)^2 + 8(10)^1 + 1(10)^0$$
$$= 100\,000\,000 + 80\,000\,000 + 28\,000\,000 + 5\,600\,000 + 700\,000 + 56\,000 + 2800 + 80 + 1$$
$$= 214\,358\,881$$

Check



b. $11^8 = (10+1)^8$

$$= {}_8C_0 (10)^8 + {}_8C_1 (10)^7 + {}_8C_2 (10)^6 + {}_8C_3 (10)^5 + {}_8C_4 (10)^4 + {}_8C_5 (10)^3 + {}_8C_6 (10)^2$$
$$+ {}_8C_7 (10)^1 + {}_8C_8 (10)^0$$

Enrichment (continued)

2. Textbook questions 1 to 3 of "Exploring Math: Chebyshev Polynomials," p. 361

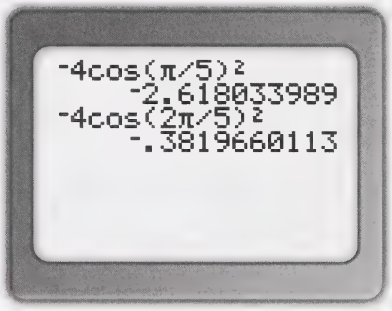
1. a. The fifth diagonal of Pascal's triangle contains the numbers 1, 3, and 1. Therefore, the Chebyshev polynomial is $x^2 + 3x + 1$.

b. $x = -4\cos^2\left(\frac{1\pi}{5}\right)$

$\doteq -2.6180$

$x = -4\cos^2\left(\frac{2\pi}{5}\right)$

$\doteq -0.3820$



The zeros are approximately $x = -2.6180$ or $x \doteq -0.3820$.

c. $a = 1$, $b = 3$, and $c = 1$

$$\therefore x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(3) \pm \sqrt{(3)^2 - 4(1)(1)}}{2(1)}$$

$$= \frac{-3 \pm \sqrt{9-4}}{2}$$

$$= \frac{-3 \pm \sqrt{5}}{2}$$

$$\therefore x = \frac{-3+\sqrt{5}}{2} \quad \text{or} \quad x = \frac{-3-\sqrt{5}}{2}$$

$$\doteq -0.3820 \qquad \doteq -2.6180$$

The zeros are approximately -0.3820 and -2.1680.

The zeros are the same for both 1.b. and 1.c.

2. a. The seventh diagonal of Pascal's triangle contains the numbers 1, 5, 6, and 1. Therefore, the Chebyshev polynomial is $x^3 + 5x^2 + 6x + 1$.

b. $x = -4\cos^2\left(\frac{1\pi}{7}\right)$

$\doteq -3.2470$

$x = -4\cos^2\left(\frac{2\pi}{7}\right)$

$\doteq -1.5550$

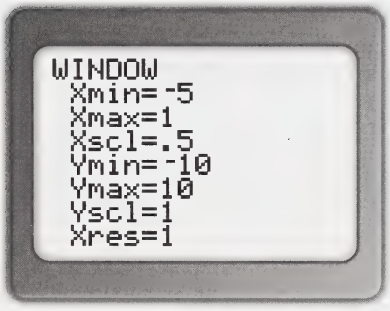
$x = -4\cos^2\left(\frac{3\pi}{7}\right)$

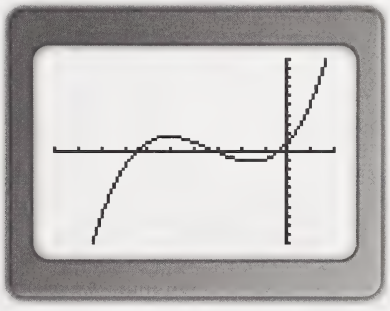
$\doteq -0.1981$

The zeros are approximately -3.2470, -1.5550, and -0.1981.

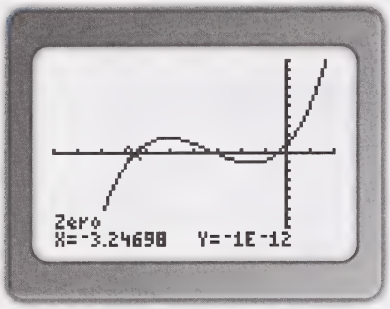
c. Use your graphing calculator to find the zeros.

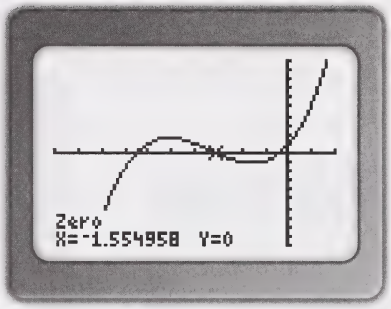
Using the window settings shown, graph $y = x^3 + 5x^2 + 6x + 1$.

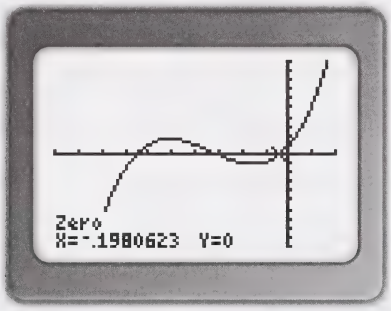




Determine the zeros of the function.







The zeros are approximately −3.2470, −1.5550, and −0.1981.

The zeros are the same in both 2.b. and 2.c.

3. A general formula for the zeros of a Chebyshev polynomial that corresponds to the *n*th diagonal is as follows:

$$x = -4\cos^2\left(\frac{k\pi}{n}\right), \text{ where } k = 1, 2, \ldots, \frac{n-1}{2}$$

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